DOCUMENT RESUME

ED 452 031 SE 064 493

AUTHOR	Nakahara, Tadao, Ed.; Koyama, Masataka, Ed.
TITLE	Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME) (24th, Hiroshima, Japan, July 23-27, 2000), Volume 1.
INSTITUTION	International Group for the Psychology of Mathematics Education.
ISSN	ISSN-0771-100X
PUB DATE	2000-00-00
NOTE	298p.; For volumes 2-4, see SE 064 994-996.
PUB TYPE	Collected Works - Proceedings (021)
EDRS PRICE	MF01/PC12 Plus Postage.
DESCRIPTORS	Algebra; Cultural Influences; *Educational Assessment; Elementary Secondary Education; Functions (Mathematics); Geometric Concepts; Geometry; Graphs; *Mathematics Curriculum; *Mathematics Education; *Problem Solving; *Statistics; Teaching Methods; Thinking Skills

ABSTRACT

The first volume of the 24th annual conference of the International Group for the Psychology of Mathematics Education includes plenary addresses, plenary panel discussions, research forum, project groups, discussion groups, short oral communications, and poster presentations. (ASK)

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ED 452 031

Proceedings

of the

24th Conference

of the

International Group for the Psychology of Mathematics Education

PME
24
2000
Hiroshima Japan

July 23-27, 2000

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Editors: Tadao Nakahara
and Masataka Koyama
Hiroshima University

Volume 1

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Volume 1

Proceedings of the 24th Conference
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Psychology of Mathematics Education

Volume 1

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ISSN 0771-100X

Logo:
The logo of the PME 24 Conference has been designed by Ayaka Kunikiyo.

Printed by the Nishiki Print Co., Ltd.
Hiroshima, Japan

PREFACE

It is an honor and pleasure for us to host the PME24 conference in Hiroshima, in the memorial year of 2000. The theme for the PME24 conference is *Major Issues in Mathematics Education for the 21st Century*. The theme for the plenary panel is *Teaching and Learning in School Mathematics*. The Program Committee hopes that the Plenary Addresses and Plenary Panel Discussion, as well as many personal presentations will create an atmosphere of reflection, examination and discussion on these significant issues.

The papers in the four volumes of the proceedings are grouped according to types of presentations: Plenary Addresses, Plenary Panel, Research Forum, Project Groups, Discussion Groups, Short Oral Communications, Poster Presentations, School Visit, and Research Reports. The plenary addresses and the research forum papers appear according to the order of presentation. The Groups are sequenced according to their numbers. For other types of presentations, within each group, papers are sequenced alphabetically by the name of the first author, with the name(s) of the presenting author(s) underlined.

There are two cross-references to help readers identify papers of interest to them:

- by research domain, according to the first author (p. 1-xxxvi);
- by author, in the list of authors (p. 1-249).

We would like to extend our thanks to the Program Committee and to the reviewers for their respective roles in working with the papers in these proceedings. We would also like to express our sincere thanks to Hideki Iwasaki, Atsumi Ueda, and Takeshi Yamaguchi for their dedication, cooperation and endless amount of work devoted to the preparation of the proceedings.

This conference received support from many sources, without which we could not have organized it to meet PME standards. We are grateful to the sponsors, especially to the Commemorative Association for the Japan World Exposition (1970), the Hiroshima Convention Bureau, and SHARP Corporation for their supports. Last, but not least, many thanks to the members of the Local Organizing Committee and many Japanese colleagues for sharing with us so willingly the responsibilites and for their personal donations given to the enterprise.

Tadao Nakahara
Masataka Koyama
Hiroshima, July 2000

TABLE OF CONTENTS

VOLUME 1

Preface 1-iii
Table of contents 1-v

Introduction

The International Group for the Psychology of Mathematics Education 1-xxix
Proceedings of previous PME conferences 1-xxxi
The review process of PME24 1-xxxiii
List of PME24 reviewers 1-xxxiv
Index of presentations by research domain 1-xxxvi

Plenary Addresses

Núñez, Rafael E. 1-3
Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics

Arzarello, Ferdinando 1-23
Inside and outside: Spaces, times, and language in proof production

Nohda, Nobuhiko 1-39
Teaching by open-approach method in Japanese mathematics classroom

Duval, Raymond 1-55
Basic issues for research in mathematics education

Plenary Panel

Theme: *Teaching and learning in school mathematics: What has research told us about mathematics teaching and learning?*

Sullivan, Peter 1-73
Physicians leave education researchers for dead

Boero, Paolo 1-76
Can research in mathematics education be useful for the teaching and learning of mathematics in school? And how?

Brown, Margaret 1-80
Does research make a contribution to teaching and learning in school mathematics? Reflections on an article from Diane Ravitch

Lin, Fou-Lai 1-84
An approach for developing well-tested, validated research of mathematics learning and teaching

Research Forum

Theme: *Dynamic geometry*
Coordinator: Laborde, Colette

Jahn, Ana Paula 1-91
New tools, new attitudes to knowledge: The case of geometric loci and transformations in Dynamic Geometry Environment

Healy, Lulu 1-103
Identifying and explaining geometrical relationship: Interactions with robust and soft Cabri constructions

Straesser, Rudolf (reactor)

Project Groups

PG1: *Intuitive rules and mathematics learning and teaching* 1-121
Coordinators: Tsamir, Pessia; Lin, Fou-Lai; Tirosh, Dina; De Bock, Dirk & Muller, Regina

PG2: *Research on mathematics teacher development* 1-122
Coordinators: Peter-Koop, Andrea & Santos-Wagner, Vânia

PG3: *Understanding of multiplicative concepts* 1-123
Coordinators: Watanabe, Tad; Pesci, Angela; Brekke, Gard & Reynolds, Anne

Discussion Groups

DG1: *Classroom research* 1-127
Coordinators: Goodchild, Simon & Linden, Nora

DG2: *Cultural aspects of the teaching and learning of mathematics* 1-128
Coordinators: Presmeg, Norma & Zack, Vicki

DG3: *Encouraging reflective practice* 1-129
Coordinators: Cockburn, Anne; Lopez-Real, Francis & Gal, Hagar

DG4: *Exploring dilemmas of research on the social aspects of mathematics education* 1-130
Coordinators: Fernandes, Elsa; Matos, João Filipe & Santos, Madalena

DG5: *Imagery and affect in mathematical learning* 1-131
Coordinators: English, Lyn D. & Goldin, Gerald A.

DG6: *Stochastical thinking, learning and teaching* 1-132
Coordinators: Truran, John; Way, Jenny; Nicholson, James & Barra, Mario

DG7: *The importance of matching research questions and methodology to the reality of researcher's lives* 1-133
Coordinators: Hart, Kath

DG8: *Theory of embodied mathematics* 1-134
Coordinators: Edwards, Laurie D. & Núñez, Rafael E.

DG9: *Underrepresented countries in PME: The state of mathematics education* 1-135
Coordinators: Denys, Bernadette & Valero, Paola

Short Oral Communications

Amaiwa, Shizuko 1-139
Relation between disposition of task and strategy of computational estimation adopted by students 10 to 17 years old

Amit, Miriam 1-140
A cross-cultural self-selection process of elementary pre-service teachers

8

Archetti, Adria; Armiento, Susanna; Basile, Eleonora; Cannizzaro, Lucilla; Crocini, Paola & Saltarelli, Lucia 1-141
The influence of the sequence of information in the solution of a word problem

Arshavsky, Nina & Goldenberg, E.Paul 1-142
Students' confusion between verbal and visual representations of geometric figures in Dynamic Geometry Environment

Barwell, Richard 1-143
Plus, and and add: Addition and English additional language learners of mathematics

Blanton, Maria L. & Kaput, James J. 1-144
Characterizing generative and self-sustaining teacher change in a classroom practice that promotes students' algebraic thinking

Borba, Rute & Nunes, T. 1-145
Are young children able to represent negative numbers?

César, Margarida 1-146
Inclusive school as a reality: The role of peer interactions in maths classes for pupils with educational special needs

Chen, Ing-Er & Lin, Fou-Lai 1-147
A thinking model of mathematics conjecturing

Cheng, Chun Chor Litwin 1-148
Assessing how children learn higher order mathematical thinking

Cohen, Nitsa 1-149
Misconceptions in 3-D geometry basic concepts

Da Costa, Nielce M. Lobo; Healy, Lulu & Magina, Sandra 1-150
Primary teachers interpretations of graphical representations teachers: Differences between static and dynamic data analysis tools

Day, Chris 1-151
The importance of practical fluency in problem solving

Denys, Bernadette & Darras, Bernard 1-152
Space as a locus of poly- and trans- disciplinarity in mathematics, art and geography

FitzSimons, Gail E.; Seah, Wee Tiong; Bishop, Alan J. & Clarkson, Philip C. 1-153
What might be learned from researching values in mathematics education?

Hannula, Markku 1-154
The metalevel of cognition-emotion interaction

Hannula, Minna 1-155
The role of tendency to focus on numerosities in the development of cardinality

Hoek, Dirk & Seegers, Gerard 1-156
Research tasks in a context of vocational education

Hoffmann, Ronit 1-157
Surprises in integrating the computer in teaching mathematics

Iida, Shinji 1-158
A survey on problem solving instruction of elemenatry school mathematics from constructivist perspective in Japan

Ishida, Junichi 1-159
What are the characteristics of the problem solving process?

Ito, Toshihiko 1-160
Gender-differences related in elementary school students' mathematics anxiety of Japan

Jahn, Ana Paula; Healy, Lulu & Campos, Tania Maria Mendonça 1-161
The devolution of Cabri activities: From researchers to teachers to students

Jess, Kristine & Andersen, Michael Wahl 1-162
The formative effect of process-oriented assessment in the mathematics teachers' practices

Kanemoto, Yoshimichi 1-163
The context in the discussion activities of mathematics classes

Kawasaki, Michihiro 1-164
Inconsistencies on the concept of geometry

Kramarski, Bracha; Mevarech, Zemira R. & Liberman, Adiva 1-165
Improving mathematical reasoning: The role of multilevel-metacognitive training

Krupanandan, Daniel 1-166
Changing teachers assessment practice through an in-service programme

Kunioka, Takahiro 1-167
Importance and difficulty of schema induction in analogical problem solving

Kuriyama, Kazuhiro & Yoshida, Hajime 1-168
Representational structure of numbers in mental addition

Lebethe, Agatha & Agherdien, Gabeba 1-169
South African mathematics teachers in transition: Making sense of the mathematics specific outcomes

Mesquita, A.L. 1-170
On the transition between space and plane

Mizoguchi, Tatsuya 1-171
Students' conceptual change of the equal symbol: On the basis of the notion of epistemological obstacle

Möller, Regina D. 1-172
Childrens' strategies to calculate prices of fruit within a real world setting

Morgan, Candia & Lerman, Steve 1-173
Teachers' positions in assessment discourses: Including a sociological perspective on the mathematics classroom

Nicholson, James & Mulhern, Gerry 1-174
Teaching and learning statistics: Diagnostic and support materials for teachers and students

Nieuwoudt, Hercules D. 1-175
Learning and teaching through problem-solving: Reflecting on two spatial cases

Nieuwoudt, Susan M. 1-176
The impact of a video class system on the teaching and learning of junior secondary school mathematics

Novotná, Jarmila & Kubínová, Marie 1-177
Two thinking strategies in word problem representation

Oguchi, Yuichi 1-178
The research of planning in open-approach method: Through the description by opportunistic planning model

Okabe, Hatsue; Hirotani, Shinji & Sakata, Hiroshi 1-179
Learning guidance for developing creative abilities and attitude: Applying the open approach method in the mathematical learning process

Okazaki, Masakazu & Iwasaki,Hideki 1-180
Geometric construction as a threshold of proof: The figure as a cognitive tool for justification

Papanastasiou, Constantinos 1-181
Effects of background and school factors on beliefs and attitudes toward mathematics

Piteira, Gisélia Correia 1-182
Geometric dynamic environment as a mediator agent for learning geometry

Santos-Wagner, Vânia Maria 1-183
Reflections about the work with teachers

Schliemann, Analúcia D.; Lara-Roth,Susanna & Epstein, Jessica 1-184
Understanding how children solve compare problems

Sklar, Jacqueline S. 1-185
College students' beliefs on number systems: An aesthetic approach to understanding

Smith, Ron 1-186
The impact of attending professional development as a group of teachers

Solomon, Jesse & Nemirovsky, Ricardo 1-187
Taking a second look

Suehiro, Satoshi 1-188
An analysis of a Japanese high school student's ability to interpret graphs

Sugaoka, Tsuyoshi 1-189
Function concepts: What college students have

Szendrei, Julianna 1-190
The difficulties of vocational school students in problem solving

Uemura, Tetsuro 1-191
A study of the features and teaching of mathematical concepts of mentally retarded children

Wake, Geoff D. & Williams, Julian S. 1-192
Daniel: A student with learning difficulties in college but competence in the workplace?

Yamaguchi, Takeshi & Iwasaki, Hideki 1-193
Separation model based on Dörfler's generalization theory

Yeap, Ban-Har & Kaur, Berinderjeet 1-194
Analyzing children's problem-posing responses

Yoshida, Hajime & Kawano, Yasuo 1-195
Informal knowledge in children before learning ratio in schools formally

Yoshida, Kaori 1-196
A study of everyday concepts and mathematical concepts based on Vygotsky's theory

Poster Presentations

Barra, Mario 1-199
Relationship between probability and other languages used in the sciences: The Pythagorean "Aritmo-geometry" expanded to d-dimensions and in the continuum and the discrete spaces. A didactic proposal

César, Margarida; Fonseca, S.; Martins, H. & Costa, Conceicao 1-200
The frog and the prince charming: Changes on pupils' social representation about maths

Gusev, Valery A. 1-201
New experimental course of geometry

Ilany, Bat-Sheva & Orbach, Haim 1-202
21 puzzle blocks: A tangible game integrated with a computer program

Iwata, Koji 1-203
What is the divergent thinking?: Four types of local divergent thinking

Kageyama, Kazuya 1-204
Features of children's spatial thinking at van Hiele's levels 1 and 2

Kajikawa, Yuji & Uehara, Shimon 1-205
Mathematics education in Germany and in Japan

Kato, Hisae 1-206
Effects of teaching for development of metacognitive ability

Kato, Ryugo 1-207
A study of the problem of teaching materials utilizing a graphical electronic calculator

Kawasaki, Michihiro 1-208
A representational model on the concept of geometry

Lebethe, Agatha & Agherdien, Gabeba 1-209
The changing mathematics curriculum in South Africa: Just another grand narrative? Or pure simulation?

Mousley, Judith A. 1-210
Moments of decision making in teaching for understanding

Nakanishi, Takashi 1-211
A study of democratic competence through mathematics education: Recommendation for the implantation of critical mathematics education in Japan

Nakano, Toshiyuki 1-212
The specific technique of custom required to negotiate mathematical meaning

Ohtani, Minoru 1-213
High context and hidden agency in Japanese mathematical discourse: A Vygotskian perspective

Safuanov, Ildar 1-214
Pupils' views of mathematics teaching in Tatarstan

Saito, Noboru & Akita, Miyo 1-215
The development of the support system for activating structural thinking

Santos-Wagner, Vânia Maria & De Carvalho, Valéria 1-216
Concept mapping and writing: Implications for continuous teacher education

Schorr, Roberta Y. & Goldin, Gerald A. 1-217
Students' interpretations of graphical representations involving changing speed

Shimada, Hideaki 1-218
Single-digit multiplication performance in Japanese adults: Assessing the problem size effect

Suwaki, Siro 1-219
Proposal to junior high school mathematics education based on "The Third International Mathematics and Science Study"

Ueda, Atsumi 1-220
Functional relations among internal representations of mathematical word problems

Way, Jenni 1-221
Enrichment for mathematically promising students in the UK

Yamagishi, Yohsuke 1-222
Case study of conception on limit concerning children

Yamaguti, Kiyoshi & Shibuya, Kenichi 1-223
Visualization of the product of complex numbers in the teaching

School Visit

Nakahara, Tadao; Fujii, Toshiakira & Koyama, Masataka 1-227
Brief Guidance to Japanese mathematics education

Lesson Plan: 1st Grade (Elementary School) 1-234

Lesson Plan: 4th Grade (Elementary School) 1-236

Lesson Plan: 5th Grade (Elementary School) 1-238

Lesson Plan: 7th Grade (Junior High School) 1-240

Lesson Plan: 9th Grade (Junior High School) 1-242

Lesson Plan: 10th Grade (Senior High School) 1-244

List of Authors 1-249

List of Sponsors 1-277

VOLUME 2

Research Reports

Aharoni, Dan 2-1
What you see is what you get: The influence of visualization on the perception of data structures

Ainley, Janet 2-9
Exploring the transparency of graphs and graphing

Askew, Mike; Brown, Margaret; Denvir, Hazel & Rhodes, Valerie 2-17
Describing primary mathematics lessons observed in the Leverhulme Numeracy Research Programme: A qualitative framework

Ayres, Paul 2-25
An analysis of bracket expansion errors

Ayres, Paul & Way, Jenni 2-33
Knowing the sample space or not: The effects on decision making

Baba, Takuya & Iwasaki, Hideki 2-41
The development of mathematics education based on ethnomathematics (2): Analysis of Universal Activities in terms of verbs

Baker, David A. & Street, B.V. 2-49
Maths as social and explanations for 'underachievement' in numeracy

Baturo, Annette R. & Cooper, Tom J. 2-57
Year 6 students' idiosyncratic notions of unitising, reunitising, and regrouping decimal number places

Baturo, Annette R.; Cooper, Tom J.; Kidman, Gillian C. & McRobbie, Campbell J. 2-65
Factors influencing teachers' endorsement of the core mathematics course of an integrated learning system

Bezuidenhout, Jan & Olivier, Alwyn 2-73
Students' conceptions of the integral

Bills, Chris & Gray, Eddie 2-81
The use of mental imagery in mental calculation

Boulton-Lewis, Gillian M.; Cooper, Tom J.; Atweh, B.; Pillay, H. & Wilss, L. 2-89
Readiness for algebra

Bragg, Philippa & Outhred, Lynne 2-97
Students' knowledge of length units: Do they know more than rules about rulers?

Breen, Chris 2-105
Becoming more aware: Psychoanalytic insights concerning fear and relationship in the mathematics classroom

Brown, Laurinda & Coles, Alf 2-113
Same/different: A 'natural' way of learning mathematics

Brown,Margaret; Denvir, Hazel; Rhodes, Valerie; Askew, Mike; Wiliam, Dylan & Ranson, Esther 2-121
The effect of some classroom factors on grade 3 pupil gains in the Leverhulme Numeracy Research Programme

Bulafo, Gildo Luís 2-129
"Automatism" in finding a "solution" among junior high school students: A comparative study

Carlson, Marilyn P. 2-137
A study of the mathematical behaviors of mathematicians: The role of metacognition and mathematical intimacy in solving problems

Carraher, David; Brizuela, Bárbara M. & Schliemann, Analúcia D. 2-145
Bringing out the algebraic character of arithmetic: Instantiating variables in addition and subtraction

Carvalho, Carolina & César, Margarida 2-153
The game of social interactions in statistics learning and in cognitive development

Cavanagh, Michael & Mitchelmore, Michael 2-161
Student misconceptions in interpreting basic graphic calculator displays

Charles, Kathy & Nason, Rod 2-169
Step skipping during the solution of partitive quotient fraction problems

Chin, Erh-Tsung & Tall, David 2-177
Making, having and compressing formal mathematical concepts

Cifarelli, Victor V. 2-185
Mental projections in mathematical problem solving: Abductive inference and schemes of action in the evolution of mathematical knowledge

Cortes, Anibal & Pfaff, Nathalie 2-193
Solving equations and inequations: Operational invariants and methods constructed by students

Czarnocha, Bronisuave & Prabhu, Vrunda 2-201
The flow of thought across the Zone of Proximal Development between elementary algebra and intermediate English as a second language

Da Rocha Falcão, Jorge Tarcísio; Brito Lima,Anna Paula; De Araújo,Cláudia Roberta; Lins Lessa, Mônica Maria & Osório, Mônica Oliveira 2-209
A didactic sequence for the introduction of algebraic activity in early elementary school

Daskalogianni, Katrina & Simpson, Adrian 2-217
Towards a definition of attitude: The relationship between the affective and the cognitive in pre-university students

Davis, Gary; Hill, David & Smith, Nigel 2-225
A memory-based model for aspects of mathematics teaching

De Bock, Dirk; Verschaffel, Lieven; Janssens, Dirk & Claes, Karen 2-233
Involving pupils in an authentic context: Does it help them to overcome the "illusion of linearity"?

Diezmann, Carmel M. 2-241
The difficulties students experience in generating diagrams for novel problems

Douek, Nadia & Scali, Ezio 2-249
About argumentation and conceptualisation

Draisma, Jan 2-257
Gesture and oral computation as resources in the early learning of mathematics

English, Lyn D.; Charles, Kathy L. & Cudmore, Donald H. 2-265
Students' statistical reasoning during a data modelling program

Forgasz, Helen J. & Leder, Gilah C. 2-273
The 'mathematics as a gendered domain' scale

Frant, Janete Bolite; Rabello de Castro, Monica & Lima, Flavio 2-281
Investigating function from a social representation perspective

Furinghetti, Fulvia & Paola, Domingo 2-289
Definition as a teaching object: A preliminary study

Gal, Hagar & Linchevski, Liora 2-297
When a learning situation becomes a problematic learning situation: The case of diagonals in the quadrangle

Greiffenhagen, Christian 2-305
From traditional blackboards to interactive whiteboards: A pilot study to inform system design

VOLUME 3

Groves, Susie; Doig, Brian & Splitter, Laurance 3-1
Mathematics classrooms functioning as communities of inquiry: Possibilities and constraints for changing practice

Guin, Dominique & Trouche, Luc 3-9
Thinking of new devices to make viable symbolic calculators in the classroom

Gusev, Valery A. & Safuanov, Ilder S. 3-17
Some theoretical problems of the development of mathematical thinking

Harada, Kouhei; Gallou-Dumiel, Elisabeth & Nohda, Nobuhiko 3-25
The role of figures in geometrical proof-problem solving: Students' cognitions of geometrical figures in France and Japan

Hardy, Tansy 3-33
Thinking about the discursive practices of teachers and children in a 'National Numeracy Strategy' lesson

Hasegawa, Junichi 3-41
Classroom discussion on the representation of quantity by fractions: Stability of misconception and implications to practice

Hino, Keiko 3-49
Process of internalizing new use of multiplication through classroom instruction: A case study

Hong, Ye Yoon; Thomas, Mike & Kwon, Oh-Nam 3-57
Understanding linear algebraic equations via super-calculator representations

Huillet, Danielle & Mutemba, Balbina 3-65
The relation of Mozambican secondary school teachers to a mathematical concept: The case of limits of functions

Igliori, Sonia, Maranhão, Cristina & Sentelhas, S. 3-71
The meaning of terms concerning the time ordering for first grade students: The influence of cultural background

Imai, Toshihiro 3-79
The relationships between fluency and flexibility of divergent thinking in open ended mathematics situation and overcoming fixation in mathematics on Japanese junior high school students

Isoda, Masami & Nakagoshi, Akemi 3-87
A case study of student emotional change using changing heart rate in problem posing and solving Japanese classroom in mathematics

Jones, Graham A.; Langrall, Cynthia W.; Thornton, Carol A.; Mooney, Edward S.; Wares, Arsalan; Perry, Bob; Putt, Ian J. & Nisbet, Steven 3-95
Using students' statistical thinking to inform instruction

Jones, Keith & Simons, Helen 3-103
The student experience of online mathematics enrichment

Kaldrimidou, Maria; Sakonidis, Haralambos & Tzekaki, Marianna 3-111
Epistemological features in the mathematics classroom: Algebra and geometry

Karsenty, Ronnie & Vinner, Shlomo 3-119
What do we remember when it's over?: Adults' recollections of their mathematical experience

Kendal, Margaret & Stacey, Kaye 3-127
Acquiring the concept of derivative: Teaching and learning with multiple representations and CAS

Kidman, Gillian C.; Baturo, Annette R.; Cooper, Tom J. & Proctor, Romina 3-135
Year 6 students' progression on the core mathematics course of an integrated learning system

Kishimoto, Tadayuki 3-143
Solving multiplicative word problems with decimal fractions: The effects of proportional reasoning and metacognition

Klein, Ronith & Tirosh, Dina 3-151
Does a research based teacher development program affect teachers' lesson plans?

Koyama, Masataka 3-159
A research on the validity and effectiveness of "two-axes process model" of understanding mathematics at elementary school level

Kramarski, Bracha 3-167
The effects of different instructional methods on the ability to communicate mathematical reasoning

Krummheuer, Götz 3-175
Narrative elements in mathematical argumentations in primary education

Kubínová, Marie; Mareš, Jiři & Novotná, Jarmila 3-183
Changing teaching methods in school mathematics: An analysis of some episodes from classes

Kumagai, Koichi 3-191
Development of concepts for division in third grade teaching experiments: From the viewpoint of the dual nature of concepts and symbolizing processes

Kutscher, Bilha & Linchevski, Liora 3-199
Moving between mixed-ability and same-ability settings: Impact on learners

Kyriakides, Leonidas 3-207
A developmental assessment of pupils use of domain-specific and general strategies in problem solving

Lawrie, Christine; Pegg, John & Gutiérrez, Angel 3-215
Coding the nature of thinking displayed in responses on nets of solids

Lim, Chap Sam 3-223
A comparison between Malaysian and United Kingdom teachers' and students' images of mathematics

Lin, Pi-Jen 3-231
On developing teachers knowledge by using cases constructed by researcher and classroom teachers

Linchevski, Liora; Kutscher, Bilha; Olivier, Alwyn & Bennie, Kate 3-239
Indispensable mathematical knowledge-IMK and differential mathematical knowledge-DMK: Two sides of the equity coin

Lowrie, Tom & Whitland, Jacquelyn 3-247
Problem posing as a tool for learning, planning and assessment in the primary school

Maranhão, Cristina & Campos,T. 3-255
Length measurement: Conventional units articulated with arbitrary ones

Markopoulos, Christos & Potari, Despina 3-263
Dynamic transformations of solids in the mathematics classroom

Matsuo, Nanae 3-271
States of understanding relations among concepts of geometric figures: Considered from the aspect of concept image and concept definition

McClain, Kay & Bowers, Janet 3-279
Supporting preservice teachers' understanding of place value and multidigit addition and subtraction

Meissner, Hartwig & Pinkernell, Guido 3-287
Spatial abilities in primary schools

Mekhmandarov, Ibby 3-295
Analysis and synthesis of the Cartesian product by kindergarten children

Merenluoto, Kaarina & Lehtinen, Erno 3-303
The "conflicting" concepts of continuity and limit: A conceptual change perspective

VOLUME 4

Miyazaki, Mikio 4-1
What are essential to apply the "discovery" function of proof in lower secondary school mathematics?

Mok, Ida Ah Chee 4-9
The anatomy of an 'open' mathematics lesson

Mulligan, Joanne & Wright, Robert (Bob) 4-17
Interview-based assessment of early multiplication and division

Murimo, Adelino Evaristo 4-25
Procedures of finding a solution for word problems: A study of Mozambican secondary school students

Neumann, Vicente; Núñez, Rafael & Mamani, Manuel 4-33
Proto-mathematical concepts in Northern Chilean Aymaras

Ninomiya, Hiroyuki 4-41
The case analysis of six graders' journal writings: Using the "framework for analyzing the quality of transactional writing"

Nisbet, Steven & Bain, John 4-49
Listen to the graph: Children's matching of melodies with their visual represenations

BEST COPY AVAILABLE

Ocaña, Lourdes Figueiras 4-57
Teaching mathematics via internet: written interactions between tutor and student

Perry, Bob; Putt, Ian J.; Jones, Graham A.; Thornton, Carol A.; Langrall, Cynthia W. & Mooney, Edward S. 4-65
Elementary school students' statistical thinking: An international perspective

Pesci, Angela 4-73
The properties of necessity and sufficiency in the construction of geometric figures with Cabri

Radford, Luis 4-81
Students' processes of symbolizing in algebra: A semiotic analysis of the production of signs in generalizing tasks

Reading, Chris & Shaughnessy, Mike 4-89
Student perceptions of variation in a sampling situation

Roddick, Cheryl; Becker, Joanne Rossi & Pence, Barbara J. 4-97
Capstone courses in problem solving for prospective secondary teachers: Effects on beliefs and teaching practices

Sackur, Catherine; Drouhard, Jean-Philippe; Maurel, Maryse & Assude, Teresa 4-105
Experiencing the necessity of a mathematical statement

Sasaki, Tetsuro 4-113
Indexicality and reflexivity in the documentary of classroom construction of ratio concept

Segalis, Bracha & Peled, Irit 4-121
The effect of mapping analogical subtraction procedures on conceptual and procedural knowledge

Sekiguchi, Yasuhiro 4-129
A teaching experiment on mathematical proof: Roles of metaphor and externalization

Shigematsu, Keiichi & Katsumi, Yoshio 4-137
Metacognition: The role of the "inner teacher" (6): Research on the relation between a transfiguration of student's mathematics knowledge and "inner teacher"

Shimizu, Norihiro 4-145
An analysis of "make an organized list" strategy in problem solving process

Shimizu, Yoshinori 4-153
Explaining your solution to younger children in a written assessment task

Simon, Martin A. 4-161
Reconsidering mathematical validation in the classroom

Skott, Jeppe 4-169
The forced autonomy of mathematics teachers

Steinbring, Heinz 4-177
The genesis of new mathematical knowledge as a social construction

Takahashi, Hitoshi 4-185
Modalities of students' internal frames of reference in learning school mathematics

Teong, Su-Kwang; Threlfall, John & Monaghan, John 4-193
The effects of metacognitive training in mathematical word problem solving in a computer environment

Teppo, Anne R. 4-201
Using algebraic processes to promote concept development

Truran, John M. 4-209
Children's learning of independence: Can research help?

Tsai, Wen Huan 4-217
Advancing arithmetic thinking based on children's cultural conceptual activities: The Pick-Red-Point game

Tsamir, Pessia & Mandel, Nurit 4-225
The intuitive rule same A - same B: The case of area and perimeter

Tsamir, Pessia & Tirosh, Dina 4-233
Intuitive beliefs and undefined operations: The cases of division by zero

Ubuz, Behiye & Kirkpinar, Burcu 4-241
Factors contributing to learning of calculus

Valero, Paola & Jess, Kristine 4-249
Supporting change through a mathematics team forum for teachers' professional development

Vermeulen, Nelis 4-257
Student teachers' concept images of algebraic expressions

Wake, Geoff D.; Williams, Julian S. & Haighton, J. 4-265
Spreadsheet mathematics in college and in the workplace: A mediating instrument?

Warren, Elizabeth 4-273
Visualisation and the development of early understanding in algebra

Winsløw, Carl 4-281
Linguistic Aspects of computer algebra systems in higher mathematics education

Yamada, Atsushi 4-289
Two patterns of progress of problem-solving process: From a representational perspective

Yates, Shirley M. 4-297
Student optimism, pessimism, motivation and achievement in mathematics: A longitudinal study

INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY AND AIMS OF PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany, in 1976. Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gerard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada) and Stephen Lerman (UK).

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators;
- To further a deeper understanding into the psychological aspects of teaching and learning mathematics and the implications thereof.

PME MEMBERSHIP AND RELATED INFORMATION

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees ($40 US or the equivalent in local currency) per year (January to December). For participants of PME24 Conference, the membership fee is included in the Conference Deposit. Others are requested to contact their Regional Contact or the Executive Secretary:

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For more information about PME as an organization see its home page at:
http://members.tripod.com/~IGPME (case sensitive) or through the Executive Secretary.

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PME International

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerpen, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, The Netherlands	ED411130(vol.1), ED411131(vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montreal, Canada	ED383532
12	1988	Veszprem, Hungary	ED411128(vol.1), ED411129(vol.2)
13	1989	Paris, France	ED411140(vol.1), ED411141(vol.2), ED411142(vol.3)
14	1990	Oaxtepex, Mexico	ED411137(vol.1), ED411138(vol.2), ED411139(vol.3)

15	1991	Assisi, Italy	ED413162(vol.1), ED413163(vol.2), ED413 6(vol.3)
16	1992	Durham, USA	ED383 538
17	1993	Tsukuba, Japan	ED383 536
18	1994	Lisbon, Portugal	ED383 537
19	1995	Recife, Brazil	ED411134(vol.1), ED411135(vol.2), ED411136(vol.3)
20	1996	Valencia, Spain	being processed
21	1997	Lahti,Finland	ED416082(vol.1), ED416083(vol.2), ED416084(vol.3), ED416085(vol.4)
22	1998	Stellenbosch, South Africa	ED427969(vol.1), ED427970(vol.2), ED427971(vol.3), ED427972(vol.4)
23	1999	Haifa, Israel	being processed

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, California (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montreal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montreal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132(vol.1), ED411 133(vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED411137(vol.1), ED411138(vol.2), ED411139(vol.3)
13	1991	Virginia (with PME16)	ED352274
14	1992	Durham, New Hampshire	ED383538
15	1993	California	ED372917
16	1994	Louisiana	ED383533(vol.1), ED383534(vol.2)
17	1995	Ohio	ED398534
18	1996	Panama City, Florida	ED400178
19	1997	Normal, Illinois	ED420494(vol.1), ED420495(vol.2)
20	1998	Raleigh, North Carolina	ED430775(vol.1), ED430776(vol.2)
21	1999	Mexico	being processed
22	2000	Tucson, Arizona	

THE REVIEW PROCESS OF PME24

Reseach Forum

Three themes had been suggested by the Program Committee as research forum themes for the PME24 conference: *Dynamic Geometry*; *Language, Semiotics and Mathematics Education*; and *Rational Numbers*. The Program Committee received only 4 research forum proposals for these themes (3 for the first theme, none for the second theme, and 1 for the third theme). For the first theme, all the proposals were reviewed and ranked by three reputable scholars with expertise in the field. The Program Committee considered and generally accepted the research forum coordinator's evaluation of the reviews and ranking of the proposals. Consequently, 3 proposals were selected for the first theme. For the second and third themes, the Program Committee had discussed thoughtfully about the disappointing situation of only one or none proposal for these themes. As a result, it was decided to cancel these forums.

Reseach Reports

The Program Committee received 159 research report proposals. Each proposal was sent for blind review to three reviewers. As a rule, proposals with at least two recommendations for acceptance were accepted. The reviews of proposals with only one recommendation for acceptance were carefully read by at least two members of the Program Committee. When necessary, the Program Committee members read the full proposal and formally reviewed it. Proposals with 3 recommendations for rejection were not considered for presentation as research reports. Altogether, 117 research report proposals were accepted. When appropriate, authors of proposals that were not accepted as research reports were invited to re-submit their work – some in the form of a short oral communication and some as a poster presentation.

Short Oral Communications and Poster Presentations

The Program Committee received 53 short oral communication proposals and 22 poster presentation proposals. Each proposal was reviewed by at least two Program Committee members. Altogether, 42 short oral proposals and 17 poster proposals were accepted. There were cases in which the Program Committee did not accept a proposal in the form that it was intended but invited the author(s) to present it in a different form.

After these reviewing procedures, the Program Committee recommended the acceptance of presentations in the following format: 117 Research Reports, 64 Short Oral Communications, and 27 Poster Presentations.

LIST OF PME24 REVIEWERS

The PME24 Program Committee thanks the following people for their help in the review process:

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Zack, Vicki (Canada)
Zaslavsky, Orit (Israel)
Zazkis, Rina (Canada)
Zehavi, Nurit (Israel)

INDEX OF PRESENTATIONS BY RESEARCH DOMAIN

The papers in the Proceedings are indexed below by research domain, mostly as indicated by the authors on their proposal form. The papers are indicated by their first author and page number.

Adult Learning

Czarnocha, B. 2-201
Lebethe, A. 1-169
Lebethe, A. 1-209
Santos-Wagner, V.M. 1-183
Smith, R. 1-186
Szendrei, J. 1-190

Advanced Mathematical Thinking

Aharoni, D. 2-1
Barra, M. 1-199
Bezuidenhout, J. 2-73
Chen, I-E. 1-147
Chin, E-T. 2-177
Gal, H. 2-297
Guin, D. 3-9
Hoffmann, R. 1-157
Iwata, K. 1-203
Merenluoto, K. 3-303
Sackur, C. 4-105
Solomon, J. 1-187
Tsai, W. H. 4-217
Ubuz, B. 4-241

Affective Factors

Breen, C. 2-105
Daskalogianni, K. 2-217
Day, C. 1-151
FitzSimons, G. E. 1-153
Isoda, M. 3-87
Ito, T. 1-160
Karsenty, R. 3-119
Kubínová, M. 3-183
Kutscher, B. 3-199
Lim, C. S. 3-223
Santos-Wagner, V.M. 1-183
Sklar, J. S. 1-185
Suwaki, S. 1-219
Takahashi, H. 4-185
Yates, S. M. 4-297

Algebraic Thinking

Amaiwa, S. 1-139
Ayres, P. 2-25
Blanton, M. L. 1-144
Boulton-Lewis, G.M. 2-89
Bulafo, G. L. 2-129
Carraher, D. 2-145
Da Rocha Falcão, J.T. 2-209
Hong, Y. Y. 3-57
Iida, S. 1-158
Kaldrimidou, M. 3-111
Kendal, M. 3-127
Murimo, A. E. 4-25
Radford, L. 4-81
Teppo, A. R. 4-201
Tsamir, P. 4-233
Vermeulen, N. 4-257
Warren, E. 4-273
Yamaguchi, T. 1-193
Yamaguti, K. 1-223

Assessment and Evaluation

Brown, M. 2-121
Cheng, C. C. L. 1-148
Day, C. 1-151
Isoda, M. 3-87
Ito, T. 1-160
Krupanandan, D. 1-166
Kyriakides, L. 3-207
Morgan, C. 1-173
Ninomiya, H. 4-41
Shimizu, Y. 4-153
Ueda, A. 1-220
Yates, S. M. 4-297

Beliefs

Amit, M. 1-140
César, M. 1-200
Daskalogianni, K. 2-217
De Bock, D. 2-233
FitzSimons, G. E. 1-153
Isoda, M. 3-87
Jones, K. 3-103
Karsenty, R. 3-119
Krupanandan, D. 1-166
Kubínová, M. 3-183
Kutscher, B. 3-199
Lebethe, A. 1-169
Lebethe, A. 1-209
Lim, C. S. 3-223
McClain, K. 3-279
Nieuwoudt, H. D. 1-175
Roddick, C. 4-97
Safuanov, I. 1-214
Sklar, J. S. 1-185
Smith, R. 1-186
Takahashi, H. 4-185
Tsamir, P. 4-233
Yates, S. M. 4-297

Computers, Calculators and Other Technological Tool
Ainley, J. 2-9
Arshavsky, N. 1-142
Baturo, A. 2-65
Cavanagh, M. 2-161
Da Costa, N. M. L. 1-150
Greiffenhagen, C. 2-305
Guin, D. 3-9
Hoek, D. 1-156
Hoffmann, R. 1-157
Hong, Y. Y. 3-57
Ilany, B-S. 1-202
Jahn, A. P. 1-161
Jones, K. 3-103
Kato, R. 1-207
Kendal, M. 3-127
Kidman, G. C. 3-135
Nieuwoudt, S. M. 1-176
Pesci, A. 4-73
Piteira, G. C. 1-182
Saito, N. 1-215
Suehiro, S. 1-188
Teong, S-K. 4-193
Ubuz, B. 4-241
Uemura, T. 1-191
Way, J. 1-221
Winsløw, C. 4-281

Data Handling(Statistics)
Ainley, J. 2-9
Da Costa, N. M. L. 1-150
Jones, G. A. 3-95
Nicholson, J. 1-174
Perry, B. 4-65
Reading, C. 4-89
Uemura, T. 1-191

Early Number Sence
Askew, M. 2-17
Baturo, A. 2-57
Bills, C. 2-81
Borba, R. 1-145
Brown, M. 2-121
Draisma, J. 2-257
Hannula, M. 1-155
Ilany, B-S. 1-202
Kishimoto, T. 3-143
Kuriyama, K. 1-168
Mekhmandarov, I. 3-295
Mulligan, J. 4-17
Shimada, H. 1-218
Takahashi, H. 4-185
Uemura, T. 1-191
Yamagishi, Y. 1-222
Yoshida, H. 1-195

Epistemology
Baba, T. 2-41
Brown, L. 2-113
Cifarelli, V. V. 2-185
Kageyama, K. 1-204
Kaldrimidou, M. 3-111
Kawasaki, M. 1-164
Kawasaki, M. 1-208
Mizoguchi, T. 1-171
Nakano, T. 1-212
Neumann, V. 4-33
Okazaki, M. 1-180
Steinbring, H. 4-177
Sugaoka, T. 1-189

Functions and Graphs
Cavanagh, M. 2-161
Frant, J. B. 2-281
Hong, Y. Y. 3-57
Huillet, D. 3-65
Kato, R. 1-207
Kendal, M. 3-127
Merenluoto, K. 3-303
Nisbet, S. 4-49
Schorr, R. Y. 1-217
Suehiro, S. 1-188
Sugaoka, T. 1-189

Gender Issues
Amit, M. 1-140
Forgasz, H. 2-273
Ito, T. 1-160

Geometrical and Spatial Thinking
Arshavsky, N. 1-142
Cohen, N. 1-149
Denys, B. 1-152
Diezmann, C. M. 2-241
Gal, H. 2-297
Gusev, V. A. 1-201
Harada, K. 3-25
Jahn, A. P. 1-161
Kageyama, K. 1-204
Kaldrimidou, M. 3-111
Kawasaki, M. 1-164
Kawasaki, M. 1-208
Lawrie, C. 3-215
Maranhão, C. 3-255
Markopoulos, C. 3-263
Matsuo, N. 3-271
Meissner, H. 3-287
Mesquita, A. L. 1-170
Nieuwoudt, H. D. 1-175
Okazaki, M. 1-180
Papanastasiou, C. 1-181
Pesci, A. 4-73
Piteira, G. C. 1-182
Sackur, C. 4-105
Tsamir, P. 4-225
Yamaguti, K. 1-223

Imagery and Visualization
Aharoni, D. 2-1
Barra, M. 1-199
Bills, C. 2-81
Cohen, N. 1-149
Kageyama, K. 1-204
Kawasaki, M. 1-164
Kawasaki, M. 1-208
Lawrie, C. 3-215
Matsuo, N. 3-271
Nisbet, S. 4-49
Novotná, J. 1-177
Okazaki, M. 1-180
Papanastasiou, C. 1-181
Warren, E. 4-273
Yamagishi, Y. 1-222
Yamaguti, K. 1-223

Language and Mathematics
Archetti, A. 1-141
Baba, T. 2-41
Barwell, R. 1-143
Czarnocha, B. 2-201
Da Rocha Falcão, J.T. 2-209
Douek, N. 2-249
Draisma, J. 2-257
Frant, J. B. 2-281
Igliori, S. 3-71
Kanemoto, Y. 1-163
Kramarski, B. 1-165
Kramarski, B. 3-167
Krummheuer, G. 3-175
Neumann, V. 4-33
Ninomiya, H. 4-41
Ohtani, M. 1-213
Santos-Wagner, V.M. 1-216
Schliemann, A. D. 1-184
Winsløw, C. 4-281
Yeap, B-H. 1-194

Mathematical Modeling
Bulafo, G. L. 2-129
Da Rocha Falcão, J.T. 2-209
De Bock, D. 2-233
Kato, R. 1-207
Mesquita, A. L. 1-170

Measurement
Bragg, P. 2-97
Maranhão, C. 3-255

Mental Models
Arshavsky, N. 1-142
Ayres, P. 2-25
Carvalho, C. 2-153
Cortes, A. 2-193
Hannula, M. 1-155
Kuriyama, K. 1-168
Mizoguchi, T. 1-171
Murimo, A. E. 4-25
Segalis, B. 4-121
Sekiguchi, Y. 4-129
Yamada, A. 4-289

Metacognition
Carlson, M. P. 2-137
Day, C. 1-151
Gusev, V. A. 3-17
Kato, H. 1-206
Kramarski, B. 1-165
Kramarski, B. 3-167
Lowrie, T. 3-247
Ninomiya, H. 4-41
Sackur, C. 4-105
Santos-Wagner, V. M. 1-216
Shigematsu, K. 4-137
Teong, S-K. 4-193
Yoshida, H. 1-195

Methods of Proof
Chin, E-T. 2-177
Furinghetti, F. 2-289
Harada, K. 3-25
Miyazaki, M. 4-1
Papanastasiou, C. 1-181
Sekiguchi, Y. 4-129
Simon, M. A. 4-161

Probability and Combinatorics
Ayres, P. 2-33
Barra, M. 1-199
Truran, J. M. 4-209

Problelm Solving
Archetti, A. 1-141
Bulafo, G. L. 2-129
Carlson, M. P. 2-137
Charles, K. 2-169
Chen, I-E. 1-147
Cifarelli, V. V. 2-185
Diezmann, C. M. 2-241
English, L. D. 2-265
Groves, S. 3-1
Harada, K. 3-25
Iida, S. 1-158
Ishida, J. 1-159

Iwata, K. 1-203
Kato, H. 1-206
Kishimoto, T. 3-143
Kunioka, T. 1-167
Kyriakides, L. 3-207
Lowrie, T. 3-247
Mok, I. A. C. 4-9
Murimo, A. E. 4-25
Novotná, J. 1-177
Oguchi, Y. 1-178
Okabe, H. 1-179
Roddick, C. 4-97
Schliemann, A. D. 1-184
Shimizu, N. 4-145
Shimizu, Y. 4-153
Szendrei, J. 1-190
Teong, S-K. 4-193
Ueda, A. 1-220
Yamada, A. 4-289
Yeap, B-H. 1-194
Yoshida, H. 1-195

Rational Numbers and Proportion
Baturo, A. 2-57
Charles, K. 2-169
De Bock, D. 2-233
Hasegawa, J. 3-41
Hino, K. 3-49
Kishimoto, T. 3-143
Klein, R. 3-151
Koyama, M. 3-159
Sasaki, T. 4-113
Yamaguchi, T. 1-193

Sicio-Cultural Studies
Amit, M. 1-140
Askew, M. 2-17
Baba, T. 2-41
Baker, D. A. 2-49
Carvalho, C. 2-153
César, M. 1-200
Draisma, J. 2-257
Hardy, T. 3-33
Igliori, S. 3-71
Kanemoto, Y. 1-163
Krummheuer, G. 3-175
Kumagai, K. 3-191
Lim, C. S. 3-223
Linchevski, L. 3-239
Maranhão, C. 3-255
Mok, I. A. C. 4-9
Morgan, C. 1-173
Nakanishi, T. 1-211
Nakano, T. 1-212
Neumann, V. 4-33
Ohtani, M. 1-213
Safuanov, I. 1-214
Sasaki, T. 4-113
Shimada, H. 1-218
Steinbring, H. 4-177
Szendrei, J. 1-190
Tsai, W. H. 4-217
Wake, G. D. 1-192
Wake, G. D. 4-265
Yoshida, K. 1-196

Teacher Education and Professional Development
Blanton, M. L. 1-144
Breen, C. 2-105
Cortes, A. 2-193
Da Costa, N. M. L. 1-150
Davis, G. 2-225
FitzSimons, G. E. 1-153
Gal, H. 2-297
Groves, S. 3-1
Hoffmann, R. 1-157
Iida, S. 1-158
Jahn, A. P. 1-161
Klein, R. 3-151
Krupanandan, D. 1-166
Lebethe, A. 1-169
Lebethe, A. 1-209
Lin, P-J. 3-231
Linchevski, L. 3-239
Lowrie, T. 3-247
McClain, K. 3-279
Morgan, C. 1-173
Mousley, J. A. 1-210
Mulligan, J. 4-17
Nicholson, J. 1-174
Nieuwoudt, H. D. 1-175
Nieuwoudt, S. M. 1-176
Roddick, C. 4-97
Santos-Wagner, V.M. 1-183
Santos-Wagner, V.M. 1-216
Skott, J. 4-169
Smith, R. 1-186
Solomon, J. 1-187
Suwaki, S. 1-219
Teppo, A. R. 4-201
Truran, J. M. 4-209
Tsai, W. H. 4-217
Valero, P. 4-249
Vermeulen, N. 4-257

Theories of Learning
Aharoni, D. 2-1
Askew, M. 2-17
Bezuidenhout, J. 2-73
Brown, L. 2-113
Chen, I-E. 1-147
Cifarelli, V. V. 2-185
Cortes, A. 2-193
Davis, G. 2-225
Douek, N. 2-249
Frant, J. B. 2-281
Gusev, V. A. 3-17
Hardy, T. 3-33
Imai, T. 3-79
Iwata, K. 1-203
Jones, G. A. 3-95
Koyama, M. 3-159
Krummheuer, G. 3-175
Kubínová, M. 3-183
Kumagai, K. 3-191
Kunioka, T. 1-167
Kutscher, B. 3-199
Lawrie, C. 3-215
Mizoguchi, T. 1-171
Mousley, J. A. 1-210

BEST COPY AVAILABLE

Mulligan, J. 4-17
Nakanishi, T. 1-211
Nakano, T. 1-212
Nieuwoudt, S. M. 1-176
Oguchi, Y. 1-178
Okabe, H. 1-179
Sasaki, T. 4-113
Sekiguchi, Y. 4-129
Suwaki, S. 1-219
Yamagishi, Y. 1-222
Yamaguchi, T. 1-193

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41

MATHEMATICAL IDEA ANALYSIS:
WHAT EMBODIED COGNITIVE SCIENCE CAN SAY ABOUT THE HUMAN NATURE OF MATHEMATICS

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ABSTRACT:
This article gives a brief introduction to a new discipline called the cognitive science of mathematics (Lakoff & Núñez, 2000), that is, the empirical and multidisciplinary study of mathematics (itself) as a scientific subject matter. The theoretical background of the arguments is based on embodied cognition, and on relatively recent findings in cognitive linguistics. The article discusses Mathematical Idea Analysis*—the set of techniques for studying implicit (largely unconscious) conceptual structures in mathematics. Particular attention is paid to everyday cognitive mechanisms such as* image schemas *and* conceptual metaphors*, showing how they play a fundamental role in constituting the very fabric of mathematics. The analyses, illustrated with a discussion of some issues of set and hyperset theory, show that it is (human) meaning what makes mathematics what it is: Mathematics is not transcendentally objective, but it is not arbitrary either (not the result of pure social conventions). Some implications for mathematics education are suggested.*

Have you ever thought why (I mean, really *why*) the multiplication of two negative numbers yields a positive one? Or why the empty class is a subclass of all classes? And why is it a class at all, if it cannot be a class of anything? And why is it unique? For most people, including mathematicians, physicists, engineers, and computer scientists, the answers to these questions have a strong dogmatic component (try these questions with your own colleagues!). It is common to encounter answers such as "well, that's the way it is", or "I don't know exactly why, but I know it works that way", and so on.

Within the culture of those who practice mathematics professionally, the dogmatic answers to these questions usually follow from definitions, axioms, and rules, they don't necessarily follow from genuine understanding. In those cases, the validation of the answer is provided by proof, not necessarily by *meaning*. This profound difference between determining *that* something is true and explaining *why* it is true, can be seen in the following historical anecdote.

Benjamin Peirce, one of Harvard's leading mathematicians in the 19th century (and the father of Charles Sanders Peirce), was once lecturing at Harvard on Euler's proof that $e^{\pi i} + 1 = 0$. In teaching this famous equation and its proof, he remarked,

> "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved

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> it, and therefore we know it must be truth." (cited in Maor, 1994, p. 160)

Of course Peirce was not the only mathematician (or mathematics teacher) to fail to understand what $e^{\pi i} + 1 = 0$ means. Even today, relatively few mathematics teachers and students understand what the equation actually *means*. Yet generation after generation of mathematics teachers and students continue to go uncomprehendingly through one version or another of Euler's proof, understanding only the regularity in the manipulations of the symbols, but not the *ideas* that make it true. This is hardly an isolated example. Meaningless truth and meaningful sense-making are fundamental components of many debates involving the nature of mathematics.

In this plenary address, I want to show that it is meaning (i.e., human meaningful ideas), what makes mathematics what it is, and that this meaning is not arbitrary, not the result of pure social conventions. My arguments will be based on contemporary embodied cognitive science. More specifically, I intend to show the following:

1. That the nature of mathematics is about human *ideas*, not just, formal proofs, axioms, and definitions (proofs, axioms, and definitions constitute only a part of mathematics, which are also realized through precise sets of ideas).
2. That these ideas are grounded in species-specific everyday cognitive and bodily mechanisms, therefore making mathematics a human enterprise, not a platonic and transcendental entity.
3. That because of this grounding, mathematical ideas are not arbitrary, that is, they are not the product of purely social and cultural conventions (although socio-historical dimensions play key roles in the formation and development of ideas).
4. That the conceptual (and idea) structure that constitutes mathematics can be studied empirically, through scientific methods.
5. That a particular methodology based on embodied cognitive science —*Mathematical Idea Analysis*—can serve this purpose.

Most of the material I will present here is based on the work I have been developing for several years in close collaboration with the cognitive linguist George Lakoff in Berkeley (Lakoff & Núñez, 1997, 1998, 2000; Núñez & Lakoff, 1998).

THE CONTEMPORARY STUDY OF IDEAS: FROM ARMCHAIR PHILOSOPHY TO SCIENTIFIC UNDERSTANDING

Throughout history, many mathematicians have tried to answer the question of the nature of meaning, truth, and ideas in mathematics. In the last century or so, various influential mathematicians, such as Dedekind, Cantor, Hilbert, Poincaré, and Weyl, to mention only a few, suggested some answers which share important elements. They all considered, in one way or another, human intuition as a fundamental starting point for their philosophical investigations: Intuitions of small integers, intuitions of collections, intuitions of movement in space, and so on (see

Dedekind, 1888/1976; Dauben on Cantor (1979); Kitcher on Hilbert (1976); Poincaré, 1913/1963; Weyl, 1918/1994). They saw these fundamental intuitions of the human mind as being stable and profound to serve as basis for mathematics.[1]

These philosophical insights tell us something important. They implicitly say that the edifice of mathematics is based on aspects of the human mind that lie *outside* of mathematics proper (i.e., these intuitions themselves are not theorems, axioms or definitions). However, beyond the philosophical and historical interest these insights may have, when seen from the perspective of nowadays' scientific standards, they present important limitations:

- First, those mathematicians were professionally trained to do mathematics, not necessarily to study ideas and intuitions. And their discipline, mathematics (as such), does not study ideas or intuitions. Today, the study of ideas (concepts and intuitions) itself is a scientific subject matter, and it is not anymore just a vague and elusive philosophical object.
- Second, the methodology they used was mainly *introspection*—the subjective investigation of one's own impressions, feelings, and thoughts. Now we know, form substantial evidence in the scientific study of intuition and cognition, that there are fundamental aspects of mental activity that are unconscious in nature and therefore inaccessible to introspection.

The moral here is that pure philosophical inquiry and introspection—although very important—give, at best, a very limited picture of the conceptual structure that makes mathematics possible. What is needed, in order to understand the nature and origin of mathematics and of mathematical meaning, is to study mathematics *itself* (with its intuitive grounding, its inferential structure, its symbol systems, etc.) as a scientific subject matter. What is needed is a cognitive science *of* mathematics, a science of *mind-based mathematics* (Lakoff & Núñez, 1997, 2000). From this perspective, the answers to these issues should be in terms of those mechanisms underlying our intuitions and ideas. That is, in terms of human cognitive, biological, and cultural mechanisms, and not in terms of axioms, definitions, formal proofs, and theorems. Let us see what important findings are helpful in providing those answers.

Embodied Cognitive Science and Recent Empirical Findings about the Nature of Mind

In recent years, there have been revolutionary advances in cognitive science—the multidisciplinary scientific study of the mind. These advances have an important bearing on our understanding of mathematics. Among the most profound of these new insights are the following:

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[1] But, they didn't think of these intuitions and basic ideas as being "rigorous" enough. This was a major reason why, later, formalism would explicitly eliminate ideas, and go on to dominate the foundational debates. Unfortunately, at that time philosophers and mathematicians didn't have the scientific and theoretical tools we have today to see that human intuitions and ideas are indeed very precise and rigorous, and that therefore the problems they were facing didn't have to do with lack of rigor of ideas and intuitions. For details, see Núñez & Lakoff, 1998, and Lakoff & Núñez, 2000).

1. *The embodiment of mind.* The detailed nature and dynamics of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.
2. *The cognitive unconscious.* Most cognitive processes is unconscious—not repressed in the Freudian sense, but simply inaccessible to direct conscious introspection. We cannot through introspection look directly at our conceptual systems and at our low-level cognitive processes. This includes most mathematical thought.
3. *Metaphorical thought.* For the most part, human beings conceptualize abstract concepts in concrete terms, using precise inferential structure and modes of reasoning grounded in the sensory motor system. The cognitive mechanism by which the abstract is comprehended in terms of the concrete is called *conceptual metaphor*[2]. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line, or space as sets of points.

In what follows I intend to give a general overview of how to apply these empirical findings to the realm of mathematical ideas. That is, while taking mathematics as a subject matter for cognitive science I will ask how certain domains in mathematics are created and conceptualized. In doing so, I will show that it is with these recent advances in cognitive science that a deep and grounded *Mathematical Idea Analysis* becomes possible (for details, see Lakoff & Núñez, 2000). Keep in mind that the major concern then is not just with *what* is true in mathematics, but with what mathematical ideas *mean*, and why mathematical truths are true by virtue of what they mean.

At this point it is important to mention that when I refer to cognitive science, I refer to contemporary embodied oriented approaches (see, for instance, Johnson, 1987; Lakoff, 1987; Varela, Thompson, & Rosch, 1991; Núñez, 1995, 1999), which are radically different from orthodox cognitive science. The latter builds on dualist, functionalist, and objectivist assumptions, while the former has explicitly denied them, especially, the mind-body split (dualism). For embodied oriented approaches any theory of mind must take into account the peculiarities of brains, bodies, and the environment in which they exist. Because of these reasons analyses of the sort I will be giving below were not even imaginable in the days of orthodox cognitive science of the disembodied mind, developed in the 1960s and early 1970s. In general, within the traditional perspective, which under the form of *neo-cognitivism* (Freeman & Núñez, 1999) is still very active today, thought is addressed in terms of the manipulation of purely abstract symbols and concepts are seen as literal — free of all biological constraints and of discoveries about the brain.

I mention this, because, unfortunately, within the mathematics education community, for many, cognitive science is synonymous with the orthodox view.

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[2] As we will see later, this is a technical term.

Because of the various limitations that this traditional view has manifested over the years, many researchers in mathematics education concerned with developmental, social, and cultural factors have rejected cognitive science as a whole, assuming that it had little to offer (Núñez, Edwards, & Matos, 1998). I want to make clear then, that Mathematical Idea Analysis comes out of embodied oriented approaches to cognitive science. For a deeper discussion of the differences between orthodox cognitive science and recent embodied oriented cognitive science, see Núñez (1997), Lakoff & Johnson (1999), and Núñez & Freeman (1999).

Ordinary Cognition and Mathematical Cognition

Substantial research in neuropsychology, child development, and animal cognition suggests that all individuals of the species *Homo Sapiens* are born with a capacity to distinguish among very small numbers of objects and events (e.g., subitizing) and to do the simplest arithmetic—the arithmetic of very small numbers (for recent reviews on these and related issues, see Dehaene, 1997, and Butterworth, 1999). These findings are important for the understanding of the biological rudiments of basic arithmetic. However, they tell us very little about the full complexity and abstraction of mathematics. There is a lot more to mathematics than the arithmetic of very small numbers. Trigonometry and calculus are very far from "3 minus 1 equals 2". Even realizing that zero is a number and that negative numbers are numbers took centuries of sophisticated development. Extending numbers to the rationals, the reals, the imaginaries, and the hyperreals requires an enormous cognitive apparatus that goes well beyond what babies and animals and a normal adult without instruction can do. So the question of the nature, origin, and meaning of mathematical ideas remains open: What are the embodied cognitive capacities that allow one to go from such innate basic numerical abilities to a deep and rich understanding of, say, college-level mathematics?

George Lakoff and I have addressed this question, using methodologies from the growing field of cognitive linguistics and psycholinguistics (more about this below). According to what we have found to date, it appears that such advanced mathematical abilities are not independent of the cognitive apparatus used outside of mathematics. Rather, it appears that the cognitive structure of advanced mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought such as image schemas, aspectual schemas, conceptual blends, and conceptual metaphor[3]. Indeed, the last one is one of the most important ones, constituting the very fabric of mathematics. It is present in all subfields of mathematics, as when we conceptualize functions as sets of points, infinite sums as having a final unique resultant state, or dynamic continuity as being static preservation of closeness (Weierstrass's ε–δ criteria).

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[3] Because of the scope of this presentation, here I will refer only to image schemas and conceptual metaphor. I will describe them in the next section.

Let us now have a look at the theoretical background of Mathematical Idea Analysis.

MATHEMATICAL IDEA ANALYSIS

Extending the study of the cognitive unconscious to mathematical cognition, implies analyzing the way in which we implicitly understand mathematics as we do it or talk about it. A large part of unconscious thought involves implicit rather than explicit, automatic, immediate understanding—making sense of things without having conscious access to the cognitive mechanisms by which we make sense of things. Ordinary everyday mathematical sense-making is not in the form of conscious proofs from axioms nor is it always the result of explicit, conscious, goal-oriented instruction. Most of our everyday mathematical *understanding* takes place without our being able to explain exactly what we understood and how we understood it. What Lakoff and I have done is to study everyday mathematical understanding of this automatic unconscious sort and to ask the following crucial questions:

- How much of mathematical understanding makes use of the same kinds of conceptual mechanisms that are used in the understanding of ordinary, nonmathematical domains?
- Are the same cognitive mechanisms used to characterize ordinary ideas also used to characterize mathematical ideas?
- If yes, what is the biological or bodily grounding of such mechanisms?

We have found that a great many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas. These include such ordinary cognitive mechanisms as those used for basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on.

Thus, for example:

- Conceptualizing the technical mathematical concept of a class makes use of the everyday concept of a collection of objects in a bounded region of space.
- Conceptualizing the technical mathematical concept of recursion makes use of the everyday concept of a repeated action.
- Conceptualizing the technical mathematical concept of complex arithmetic makes use of the everyday concept of rotation.
- Conceptualizing derivatives in calculus requires making use of such everyday concepts as motion, approaching a boundary, and so on.

From a nontechnical perspective, this should be completely obvious. But from the technical perspective of cognitive science, there is a challenging question one must ask:

> *Exactly what everyday concepts and cognitive mechanisms are used in exactly what ways in the unconscious conceptualization of technical ideas, such that they provide the precise inferential structure observed in mathematics?*

Mathematical Idea Analysis, depends crucially on the answers to this question. We have found that mathematical ideas, are grounded in bodily-based mechanisms and everyday experience. Many mathematical ideas are ways of mathematicizing ordinary ideas, as when the idea of subtraction mathematizes the ordinary idea of distance, or as when the idea of a derivative mathematicizes the ordinary idea of instantaneous change. I will illustrate these findings in more detail with some examples taken from set theory and hyperset theory. But because of the technicalities involved we must first go over some basic notions of cognitive linguistics, necessary to understand those examples.

Some Basic Notions of Cognitive Linguistics and the Embodied Mind

Recent developments in cognitive linguistics have been very fruitful in studying high-level cognition from an *embodiment* perspective (e.g., natural language understanding and conceptual systems). In particular, cognitive semantics (Sweetser, 1990, Talmy, 1999), conceptual integration (Fauconnier, 1997; Fauconnier & Turner, 1998) and conceptual metaphor theory (Lakoff, 1993; Lakoff & Johnson, 1980, 1999; Gibbs, 1994) have proven to be very powerful. These approaches offer the possibility of empirically studying the conceptual structure of vast systems of abstract concepts through the largely unconscious, effortless, everyday linguistic manifestations. They provide an excellent background for the development of Mathematical Idea Analysis.

Conceptual metaphor

An important finding in cognitive linguistics is that concepts are systematically organized through vast networks of *conceptual mappings*, occurring in highly-coordinated systems and combining in complex ways. For the most part these conceptual mappings are used unconsciously and effortlessly in everyday communication. An important kind of mapping is the one mentioned earlier, *conceptual metaphor*.

It is important to keep in mind that conceptual metaphors are not mere figures of speech, and that they are not just pedagogical tools used to illustrate some educational material. Conceptual metaphors are in fact fundamental cognitive mechanisms (technically, they are *inference-preserving cross-domain mappings*) which project the inferential structure of a *source domain* onto a *target domain*, allowing the use of effortless species-specific body-based inference to structure abstract inference. For example, humans naturally conceptualize Time (target domain) primarily in terms of Uni-dimensional Motion (source domain), either the motions of future times toward an observer (as in "Christmas is *approaching*") or the motion of an observer over a time landscape (as in "We're *approaching* Christmas"). That is, our everyday concept of Time is inextricably related to the experience of uni-dimensional motion. There are, of course, many more important details and variations of this general Time As Motion mapping but their analyses would go beyond the scope of this presentation. The point here, is that these conceptual metaphors (and conceptual mappings in general) are irreducible, they are extremely precise (e.g., in the Time As Motion example, their inferential structure preserves transitive relations), they are used extensively, effortlessly, unconsciously, and they are ultimately bodily grounded (for details, see Lakoff & Johnson, 1999, Chapter 10; Núñez, 1999).

Contrary to what some people think, conceptual metaphors (and conceptual mappings in general) are not mere arbitrary social conventions. They are not arbitrary, because they are structured by species-specific constrains underlying our everyday experience—especially bodily experience. For example, in most cultures Affection is conceptualized in terms of thermic experience: Warmth (as in "He greeted me *warmly*", or as in "send her my *warm* helloes"). The grounding of this mapping doesn't depend (only) on social conventions. It emerges from the correlation all individuals of the species experience, from early ontogenetic development, between affection and the bodily experience of warmth. It is also important to mention that a huge amount of the conceptual metaphors we use in everyday communication, such as Affection Is Warmth, is not learned through explicit goal-oriented educational intervention.

Research in contemporary conceptual metaphor theory indicates that there is an extensive conventional system of conceptual metaphors in every human conceptual system. As I said earlier, unlike traditional studies of metaphor, contemporary embodied views *don't see conceptual metaphors as residing in words, but in thought.* Metaphorical linguistic expressions thus are only surface manifestations of metaphorical thought. These theoretical claims are based on substantial empirical evidence from a variety of sources, including among others, psycholinguistic experiments (Gibbs, 1994), cross-linguistic studies (Yu, 1998), generalizations over inference patterns (Lakoff, 1987), generalizations over conventional and novel language (Lakoff and Turner, 1989), the study of historical semantic change (Sweetser, 1990), of language acquisition (C. Johnson, 1997), of spontaneous gestures (McNeill, 1992), and of American sign language (Taub, 1997). Conceptual mappings thus can be studied empirically, and stated precisely.

In what concerns mathematical concepts, Lakoff & Núñez (2000) distinguish, three important types of conceptual metaphors:

- *Grounding metaphors*, which ground our understanding of mathematical ideas in terms of everyday experience. In these cases, the target domain of the metaphor is mathematical, but the source domain lies outside of mathematics. Examples include the metaphor Classes Are Container Schemas (see below) and other conceptual metaphors for arithmetic.
- *Redefinitional metaphors*, which are metaphors that impose a technical understanding replacing ordinary concepts (such as the conceptual metaphor used by Georg Cantor to reconceptualize the notions of "more than" and "as many as" for infinite sets).
- *Linking metaphors*, which are metaphors within mathematics itself that allow us to conceptualize one mathematical domain in terms of another mathematical domain. In these cases, both domains of the mapping are mathematical. Examples include Von Neumann's Numbers Are Sets metaphor, Functions Are Sets of Points, and as we will see later, the Sets Are Graphs metaphor.

The linking metaphors are in many ways the most interesting of these, since they are part of the fabric of mathematics itself. They occur whenever one branch of mathematics is used to model another, as happens frequently. Moreover, linking metaphors are central to the creation, not only of new mathematical concepts, but often to the creation of new branches of mathematics. Such classical branches of mathematics as analytic geometry, trigonometry, and complex analysis owe their existence to linking metaphors.

Spatial relation concepts and image schemas

Another important finding in cognitive linguistics is that conceptual systems can be ultimately decomposed into primitive spatial relations concepts called *image schemas*. Image schemas are basic dynamic topological and orientation structures that characterize spatial inferences and link language to visual-motor experience (Johnson, 1987; Lakoff and Johnson, 1999). As we will see, an extremely important feature of image schemas is that their *inferential structure* is *preserved* under metaphorical mappings. Image schemas can be studied empirically through language (and spontaneous gestures), in particular through the linguistic manifestation of spatial relations.

Every language has a system of spatial relations, though they differ radically from language to language. In English there are prepositions like *in, on, through, above,* and so on. Other languages have systems that often differ radically from the English system. However, the spatial relations in a given language decompose into conceptual primitives (image schemas) that appear to be universal.

For example, the English word "on," in the sense used in "The book is *on* the desk" is a composite of three primitive image schemas:

- The *Above Schema* (the book is *above* the desk)
- The *Contact Schema* (the book is *in contact* with the desk)
- The *Support Schema* (the book is *supported* by the desk)

The Above Schema is orientational; it specifies an orientation in space relative to the gravitational pull one feels on one's body. The Contact Schema is one of a number of topological schemas; it indicates an absence of a gap. The Support Schema is force-dynamic in nature; it indicates the direction and nature of a force. In general, static image schemas fall into one of these categories: orientational, topological, and force-dynamic. In other languages, the primitives may combine in very different ways. Not all languages have a single concept like English *on*. For instance, even in a language as close as German, the *on* in *on the table* is rendered as *auf*, while the *on* in *on the wall* (which does not contain the Above Schema) is translated as *an*.

A common image schema that is of great importance in mathematics is the *Container Schema*, which in everyday cognition occurs as the central part of the meaning of words like *in* and *out*. The Container Schema has three parts: an Interior, a Boundary, and an Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides, in this case an Inside and an Outside. This structure is topological in the sense that the boundary can be made larger, smaller, or distorted and still remain the boundary of a Container Schema.

The schemas for the concepts In and Out, have a bit more structure than the plain Container Schema. The concept In requires that the Interior of the Container Schema be *profiled*, that is, that it must be highlighted over the Exterior and Boundary. In addition, there is also a figure-ground distinction. For example, in a sentence like "The car is in the garage," the garage is the ground, that is, it is the landmark relative to which the car, the figure, is located. In cognitive linguistics, the ground in an image schema is called the *Landmark*, and the figure is called the *Trajector*. Thus, the In-Schema has the following structure:

- Container Schema, with Interior, Boundary, and Exterior
- Profiled: The Interior
- Landmark: The Interior

Image schemas have a special cognitive function: they are both *per*ceptual and *con*ceptual in nature. As such, they provide a bridge between language and reasoning on the one hand and vision on the other. Image schemas can fit visual perception, as when we see the milk as being *in* the glass. They can also be imposed on visual scenes, as when we see the bees swarming *in* the garden, where there is no physical container that the bees are in. Because spatial relations terms in a given language name complex image schemas, image schemas are the link between language and spatial perception.

We can now analyze how the inferential structure of image schemas (for example, the Container Schema) is preserved under metaphorical mappings to generate more abstract concepts (such as the concept of Boolean class). We shall see exactly how image schemas provide the inferential structure to the source domain of the conceptual metaphor, which via the mapping is projected onto the target domain of the metaphor to generate Boolean-class inferences.

Image schema structure and metaphorical projections

When we draw illustrations of Container Schemas, we find that they look rather like Venn Diagrams for Boolean classes. This is by no means an accident. The reason is that classes are normally conceptualized in terms of Container Schemas. For instance, we think (and speak) of elements as being *in* or *out* of a class. Venn Diagrams are visual instantiations of Container Schemas. The reason that Venn diagrams work as symbolizations of classes is that classes are usually metaphorically conceptualized as containers — that is, as bounded regions in space.

Container Schemas have a logic that appears to arise from the structure of our visual and imaging system, adapted for more general use. More specifically, Container Schemas appear to be realized neurally using such brain mechanisms as topographic maps of the visual field, center-surround receptive fields, gating circuitry, and so on (Regier, 1996). The inferential structure of these schemas can be used both for structuring space and for more abstract reason, and is projected onto our everyday conceptual system by a particular conceptual metaphor, the Categories (or 'Classes') Are Containers metaphor. This accounts for part (by no means all!) of our reasoning about conceptual categories. Boolean logic also arises from our capacity to perceive the world in terms of container schemas and to form mental images using them.

So, how do we normally conceptualize the intuitive premathematical notion of classes? The answer is in terms of Container Schemas. In other words, we normally conceptualize a class of entities in terms of a bounded region of space, with members of the class all *inside* the bounded region and nonmembers *outside* of the bounded region. From a cognitive perspective, intuitive classes are thus metaphorical conceptual containers, characterized cognitively by a metaphorical mapping —a grounding metaphor — the Classes Are Containers Schemas metaphor. The following is the mapping of such conceptual metaphor.

Classes Are Containers

Source Domain Container Schemas		Target Domain Classes
Interiors Of Container Schemas	→	Classes
Objects in Interiors	→	Class members
Being an Object in an Interior	→	The Membership Relation
An Interior of one Container Schema within a Larger One	→	A subclass in a Larger Class
The Overlap of the Interiors of Two Container Schemas	→	The Intersection of Two Classes
The Totality of the Interiors of Two Container Schemas	→	The Union of Two Classes
The Exterior of a Container Schemas	→	The Complement of a Class

This is our natural, everyday unconscious conceptual metaphor for what a class is. It is a *grounding metaphor*. It grounds our concept of a class in our concept of a bounded region in space, via the conceptual apparatus of the image schema for containment. This is the way we conceptualize classes in everyday life.

We can now analyze, how conceptual image schemas (in this case, Container Schemas) are the source of four fundamental inferential laws of logic. The structural constraints on Container Schemas mentioned earlier (i.e., brain mechanisms such as topographic maps of the visual field, center-surround receptive fields, gating circuitry, etc.) give them an inferential structure, which Lakoff and I called "Laws of Container Schemas" (Lakoff & Núñez, 2000). These so-called "laws" are conceptual in nature and are reflections at the cognitive level of brain structures at the neural level (see Figure 1). The four inferential laws are Container Schema versions of classical logical laws: *Excluded Middle*, *Modus Ponens*, *Hypothetical Syllogism*, and *Modus Tollens*. Let's see the details.

Inferential Laws of Embodied Container Schemas:

- *Excluded Middle*. Every object *X* is either *in* Container Schema *A* or *out of* Container Schema *A*.

- *Modus Ponens*: Given two Container Schemas *A* and *B* and an object *X*, if *A* is *in B* and *X* is *in A*, then *X* is *in B*.
- *Hypothetical Syllogism*: Given three Container Schemas *A*, *B* and *C*, if *A* is *in B* and *B* is *in C*, then *A* is *in C*.
- *Modus Tollens*: Given two Container Schemas *A* and *B* and an object *Y*, if *A* is *in B* and *Y* is *outside of B*, then *Y* is *outside of A*.

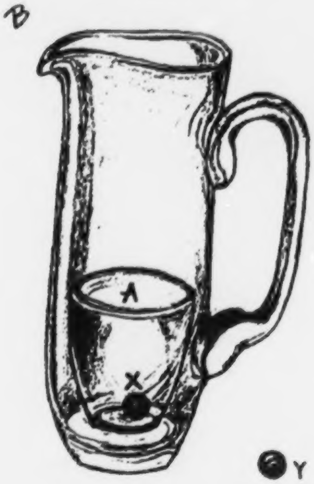

Figure 1. The "laws" of cognitive Container Schemas. The figure shows one cognitive Container Schema, *A*, occurring inside another, *B*. By inspection, one can see that if *X* is in *A*, then *X* is in *B*, and that if *Y* is outside of *B*, then *Y* is outside of *A*. We conceptualize physical containers in terms of cognitive containers. Cognitive Container Schemas are used not only in perception and imagination but also in conceptualization, as when we conceptualize bees as swarming *in* the garden. Container Schemas are the cognitive structures that allow us to make sense of familiar Venn diagrams.

Now, recall that conceptual metaphors allow the inferential structure of the source domain to be used to structure the target domain. So, the Classes Are Containers Metaphor maps the inferential laws given above for embodied Container Schemas (source domain) onto conceptual classes (target domain). These include both everyday classes and Boolean classes, which are metaphorical extensions of everyday classes. The entailment of such conceptual mapping is the following:

Inferential Laws for Classes Mapped from Embodied Container Schemas

- *Excluded Middle*. Every element *X* is either a *member of* class *A* or *not a member of* class *A*.
- *Modus Ponens*: Given two classes *A* and *B* and an element *X*, if *A* is a *subclass B* and *X* is a *member of A*, then *X* is a *member of B*.
- *Hypothetical Syllogism:* Given three classes *A*, *B*, and *C*, if *A* is a *subclass* of *B* and *B* is a *subclass of C*, then *A* is a *subclass of C*.

- *Modus Tollens*: Given two classes *A* and *B* and an element *Y*, if *A* is a *subclass of B* and *Y* is *not a member of B*, then *Y* is *not a member of A*.

The moral then is that these traditional laws of logic are in fact cognitive entities and, as such, are grounded in the neural structures that characterize Container Schemas. In other words, these laws are part of our bodies. Since they do not transcend our bodies, they are not laws of any transcendent reason. The truths of these traditional laws of logic are thus not dogmatic. They are true by virtue of what they mean.

This completes our brief and general overview of some crucial concepts of cognitive linguistics. Let us now see how this background can be used to apply a Mathematical Idea Analysis to some specific mathematical domains, Set theory and Hyperset theory.

ARE HYPERSETS, SETS?
A VIEW FROM MATHEMATICAL IDEA ANALYSIS

Consider the following question in modern mathematics: Are hypersets, sets? If not, what are they? We will now see, what embodied cognitive science can say about this. Since hypersets and sets are human (technical, mathematical) ideas we can provide an answer through Mathematical Idea Analysis. This is what we can say.

Sets

On the formalist view of the axiomatic method, a "set" is any mathematical structure that "satisfies" the axioms of set theory as written in symbols. The traditional axioms for set theory (the Zermelo-Fraenkel axioms) are often taught as being about sets conceptualized as containers. Many writers speak of sets as "containing" their members, and most students think of them that way. Even the choice of the word "member" suggests such a reading, as do the Venn diagrams used to introduce the subject. But if you look carefully through those axioms, you will find nothing in them that characterizes a container. The terms "set" and "member of" are both taken as undefined primitives. In formal mathematics, that means that they can be anything that fits the axioms. Here are the classic Zermelo-Fraenkel axioms, including the axiom of choice, what are commonly called the ZFC axioms.

The axiom of extension: Two sets are equal if and only if they have the same members. In other words, a set is uniquely determined by its members.
The axiom of specification: Given a set *A* and a one-place predicate, $P(x)$ that is either true or false of each member of *A*, there exists a subset of *A* whose members are exactly those members of *A* for which $P(x)$ is true.
The axiom of pairing: For any two sets, there exists a set that they are both members of.
The axiom of union: For every collection of sets, there is a set whose members are exactly the members of the sets of that collection.
The axiom of powers: For each set *A*, there is a set $P(A)$ whose members are exactly the subsets of set *A*.

The axiom of infinity: There exists a set A such that (1) the empty set is a member of A, and (ii) if x is a member of A, then the successor of x is a member of A.

The axiom of choice: Given a disjointed set S whose members are nonempty sets, there exists a set C which has as its members one and only one element from each member of S.

You can see that there is absolutely nothing in these axioms that explicitly requires sets to be containers. What these axioms do, collectively, is to *create* entities called "sets," first from elements and then from previously created sets. The axioms do not say explicitly how sets are to be conceptualized.

The point here is that, within formal mathematics, where all mathematical concepts are mapped onto set-theoretical structures, the "sets" used in these structures are not technically conceptualized as the Container Schemas we described above. They do not have container-schema structure with an interior, boundary, and exterior at all. Indeed, within formal mathematics, there are no concepts at all, and hence sets are not conceptualized as anything in particular. They are undefined entities whose only constraints are that they must "fit" the axioms. For formal logicians and model theorists, sets are those entities that fit the axioms and are used in the modeling of other branches of mathematics.

Of course, most of us do conceptualize sets in terms of Container Schemas, and that is perfectly consistent with the axioms given above. However, when we conceptualize sets as Container Schemas, a particular entailment follows automatically: *Sets cannot be members of themselves*, since containers cannot be inside themselves. But strictly speaking, this entailment does not follow from the axioms themselves, but rather from our metaphorical understanding of sets in terms of containers. The above axioms do not rule out sets that contain themselves. Indeed, an extra axiom was proposed by Von Neumann to rule out this possibility:

The Axiom of Foundation: There are no infinite descending sequences of sets under the membership relation. That is, $\ldots S_{i+1} \in S_i \in \ldots \in S$ is ruled out.

Since allowing sets to be members of themselves would result in such a sequence, this axiom has the indirect effect of ruling out self-membership.

Hypersets

Technically within formal mathematics, model theory has nothing to do with everyday understanding. Model-theorists do not depend upon our ordinary container-based concept of a set. Indeed, certain model-theorists have found that our ordinary grounding metaphor that Classes Are Container Schemas gets in the way of modeling kinds of phenomena they want to model, especially recursive phenomena. For example, take expressions like

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}$$

If we observe carefully, we can see that the denominator of the main fraction has in fact the value defined for x itself. In other words the above expression is equivalent to

$$x = 1 + \frac{1}{x}$$

Such recursive expressions are common in mathematics and computer science. The possibilities for modeling such expressions using "sets" are ruled out if the only kind of "sets" used in the modeling must be ones that cannot have themselves as members. Set-theorists have realized that a new non-container metaphor is needed for thinking about sets, and have explicitly constructed one (see Barwise and Moss, 1991).

The idea is to use graphs, not containers, for characterizing sets. The kinds of graphs used are Accessible Pointed Graphs, or APGs. "Pointed" indicates an asymmetric relation between nodes in the graph, indicated visually by an arrow pointing from one node to another—or from one node back to that node itself (see Figure 2). "Accessible" indicates that there is a single node which is linked to all other nodes in the graph, and can therefore be "accessed" from any other node.

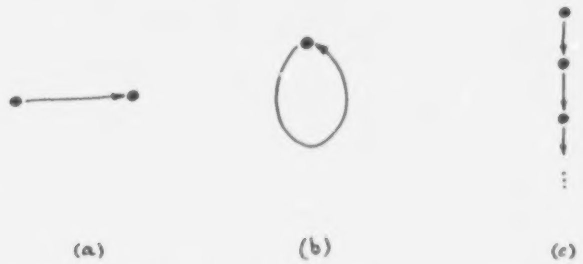

Figure 2. Hypersets: Sets conceptualized as graphs, with the empty set as the graph with no arrows leading from it. The set containing the empty set is a graph whose root has one arrow leading to the empty set (a). Illustration (b) depicts a graph of a set that is a "member" of itself, under the Sets Are Graphs Metaphor. Illustration (c) depicts an infinitely long chain of nodes in an infinite graph, which is equivalent to (b).

From the axiomatic perspective, they have replaced the Axiom of Foundation with another axiom that implies its negation, the "Anti-Foundation Axiom." From the perspective of Mathematical Idea Analysis they have implicitly used a conceptual metaphor, a linking metaphor whose mapping is the following:

The Sets Are Graphs Metaphor

Source Domain Accessible Pointed Graphs		Target Domain Sets
An APG	→	The Membership Structure of a Set
An Arrow	→	The Membership Relation
Nodes That Are Tails of Arrows	→	Sets
Decorations on Nodes that are Heads of Arrows	→	Members
APG's With No Loops	→	Classical Sets With The Foundation Axiom
APG's With or Without Loops	→	Hypersets With the Anti-Foundation Axiom

The effect of this metaphor is to eliminate the notion of containment from the concept of a "set." The graphs have no notion of containment built into them at all. And containment is not modeled by the graphs.

Graphs that have no loops satisfy the ZFC axioms and the Axiom of Foundation. They thus work just like sets conceptualized as containers. But graphs that do have loops model sets that can "have themselves as members." They do not work like sets that are conceptualized as containers, and they do not satisfy the Axiom of Foundation.

A "hyperset" is an APG that may or may not contain loops. Hypersets thus do not fit the Axiom of Foundation, but rather another axiom with the opposite intent:

- *The Anti-Foundation Axiom*: Every APG pictures a unique set.

The fact that hypersets satisfy the Zermelo-Fraenkel axioms confirms what we said above: *The Zermelo-Fraenkel axioms for set theory—the ones generally accepted in mathematics—do not define our ordinary concept of a set as a container at all!* That is, the axioms of "set theory" are not, and were never meant to be, about what we ordinarily call "sets", which we conceptualize in terms of Container Schemas.

So, What are sets, really?

Here we see the power of conceptual metaphor in mathematics. Sets, conceptualized in everyday terms as containers, do not have the right properties to model everything needed. So we can now metaphorically reconceptualize "sets" to exclude containment by using certain kinds of graphs. The only confusing thing is that this special case of graph theory is still called "set theory" for historical reasons.

Because of this misleading terminology, it is sometimes said that the theory of hypersets is "a set theory in which sets can contain themselves." From a cognitive point of view this is completely misleading because it is not a theory of "sets" as we ordinarily understand them in terms of containment. The reason that these graph theoretical objects are called "sets" is a functional one: they play the role in modeling axioms that classical sets with the Axiom of Foundation used to play.

The moral is that mathematics has (at least) two quite inconsistent metaphorical conceptions of sets, one in terms of Container Schemas (a grounding metaphor) and one in terms of graphs (a linking metaphor). Is one of these conceptions right and the other wrong? There is a perspective from which one might think so, a perspective that says that there must be only one literal correct notion of a "set". But from the perspective of Mathematical Idea Analysis these two distinct notions of "set" define different and mutually inconsistent subject matters, conceptualized via radically different conceptual metaphors. This situation is much more common in mathematics than the general public normally recognizes. It is Mathematical Idea Analysis that helps us to see and analyze these situations, by making explicit what is cognitively implicit.

EPILOGUE: SOME SPECULATIONS ABOUT THE IMPLICATIONS OF MATHEMATICAL IDEA ANALYSIS FOR MATHEMATICS EDUCATION

I would like to close my presentation, making some general remarks about possible implications of Mathematical Idea Analysis for mathematics education in general, and for the psychology of mathematics education in particular. This is by no means an exhaustive list. It is simply an open list to be taken as a proposal for discussion during the various sessions of the PME-2000 meeting.

In a nutshell, I could say that the deepest implication that Mathematical Idea Analysis provides, is the kind of philosophy of mathematics and of mathematics education that it brings forth. The approach presented here gives a portrait of mathematics that is fundamentally human. Concepts and ideas are human, and the truths that come out of them are relative to human conceptual systems. This includes mathematics. It follows from this perspective that teaching mathematics implies teaching human meaning, and teaching *why* theorems are true by virtue of what the elements involved actually mean. From this perspective at least the following implications can be mentioned:

- Mathematics education should demystify truth, proof, definitions, and formalisms. Although they are relevant, they should be taught in the context of the underlying human ideas. Therefore questions like those in the first paragraph of this article should be taken *very* seriously in the educational process.
- Mathematics Education should also demystify the belief that meaning, intuition, and ideas are vague and (purely) subjective. Human ideas and meaning have an impressive amount of bodily grounded constrains that make them non-arbitrary.
- Mathematics should be taught as a human enterprise, with its cultural and historical dimensions (which shouldn't just be a presentation of dates and a chronological list of events). These human dimensions should include those moments of doubts, hesitations, triumphs, and insights that shaped the historical process of sense-making.
- New generations of mathematics teachers, not only should have a good background in education, history, and philosophy, but they should also have some knowledge of cognitive science, in particular of the empirical study of conceptual structures and of everyday unconscious inferential mechanisms. They should know what is the implicit conceptual structure of the ideas they have to teach.
- The so-called "misconceptions" are not really misconceptions. This term as it is implies that there is a "wrong" conception, wrong relative to some "truth". But Mathematical Idea Analysis shows that there are no wrong conceptions as such, but rather variations of ideas and conceptual systems with different inferential structures (sometimes even inconsistent with each other, as we saw for the case of sets and hypersets).
- From a pedagogical point of view then, it would be very important to identify what are exactly the variations of inferential structure that generate the so-called misconceptions. By making this explicit, a pedagogical intervention should follow

for inducing students to operate with the appropriate conceptual mappings that bring forth the inferential structure required by the mathematical idea in question.

- When applied properly, Mathematical Idea Analysis can serve as a tool for helping people (especially adolescents and adults) to become aware of the organization, limitations, and potentials of their own conceptual systems, making explicit (and conscious) what in everyday life is implicit (and unconscious).
- "Being good at mathematics" doesn't necessarily mean being good at doing calculations and running algorithms. It means knowing how to keep one's metaphors straight, when to operate with the appropriate metaphors, when to shift from one to another one, when to combine them, and so on. Teaching how to master this conceptual gymnastics should be a goal for mathematics education.
- Beyond the mathematical content as such, the empirical study of conceptual systems can also give important insights into the attitudes and beliefs, students have about mathematics. The detailed study of students' conceptual structures underlying their linguistic expressions can reveal the origin of difficulties, lack of motivation, anxieties, and so on, that may be interfering with the learning of mathematics.

As you can see, this is far from being an exhaustive list. I believe that the cognitive science of mathematics, and Mathematical Idea Analysis in particular, provide a rich and deep tool, with a solid theoretical background, for bringing back human meaning into mathematics. The invitation is then extended for exploring how this can be accomplished in the process of teaching this astonishing conceptual structure called mathematics.

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INSIDE AND OUTSIDE: SPACES, TIMES AND LANGUAGE IN PROOF PRODUCTION

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Abstract. *The paper focuses on some cognitive and didactical phenomena which feature processes and products of pupils (grades 7-12), who learn 'mathematical proof' within technological environments. Language and Time reveal crucial and assume specific features when subjects interact with artefacts and instruments, because of the semiotic mediation by precise interventions of the teacher. The main issues in the analysis of students' performances consist in metaphors, deictics, mental times, narratives, functions of dragging, abductions, linear vs. multivariate language and so on, to be used within an embodied cognition perspective. The learning of proof is described as a long process of interiorisation, through specific and complex mental dynamics of pupils, from perceptions and actions within technological environments towards structured abstract mathematical objects, embedded in a theoretical framework.*

Introduction

The purpose of this paper is to focus on the genesis of (abstract) mathematical objects within specific mathematical areas, when pupils interact with technological artefacts and instruments (for the distinction see: Bartolini Bussi & Mariotti, 1999a), in particular the computer. The term 'mathematical object' is used here in a wide sense, which covers *concepts* (including representations: see Vergnaud, 1990) as well as *relationships* among them: we call *structured mathematical object*[1] such a cluster and use the abbreviation SMO to denote it. An example of SMO is the set of natural numbers (represented, let us say, in base ten): it is a set equipped with the function of successor, the usual operations and their properties.

More specifically, the paper sketches a theoretical framework, where some major variables in the genesis of SMOs are described and scrutinised; the model is based on the analysis of processes and products in pupils (aged 12 to 18) who approach 'mathematical proof' in the classroom[2]. In some cases, students work within

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(°) The results are based on a joint research, funded by the Italian Ministry of the University (MURST), which involves many Italian researchers (all their presentations at the last PME's have been obtained within such a project) and that I co-ordinate. In particular the following people have contributed to the theoretical elaboration as well as to the experimental work: V.Andriano, P.Boero, M.G.Bartolini Bussi, G.Gallino, R.Garuti, M.A.Mariotti, M.Maschietto, F.Olivero, D.Paola, O.Robutti. A special acknowledgement is due to P.Boero, M.G.Bartolini Bussi, F.Olivero, D.Paola and O.Robutti for their useful criticism and suggestions in the writing of the manuscript.

[1] SMOs have some similarity with the way mathematical entities are presented in category theory, namely as objects equipped with arrows and represented through diagrams.

[2] In the whole, about 100 students were involved in our research: they have been observed and video-taped while exploring situations, conjecturing hypotheses, proving properties (generally in peer or group interaction); moreover their written protocols (sometimes individual, sometimes not) were studied. The pupils belong to various schools in different parts of Northern Italy, where the research-teachers of our project work. Generally, pupils of 14-18 years attend the Liceo Scientifico (that is a high school with a scientific option), whilst the youngest (12-13 y.) attend the last years of the junior secondary school.

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computer-based environments (e.g. Cabri-Géomètre) or interact with other artefacts (e.g. Mathematical Mechanisms), in other cases they simply use paper and pencil; the mathematical areas exploited are Geometry and (elementary) Number Theory.

The genesis of SMOs will be pictured as a dynamic evolution from *perceptions*, *deictics*[3], *actions* to *mathematical objects, symbols, relationships* through a complex setting of *transformational processes*, which involve two main tools or categories:

(*i*) *language* (in its different forms: body l., oral l., written l.), by means of which subjects start, develop and support the whole genetic process;

(*ii*) *time*, as a psychological category which represents the mental environment where the genetic process of SMOs 'is born' and 'lives'.

The paper is divided into three sections: in *Section 1* the theoretical framework is discussed; in *Section 2* time and language are analysed with respect to the mental processes of pupils who are constructing SMOs; in *Section 3* an emblematic case study is discussed; some provisional conclusions are sketched at the end of the paper.

1. The theoretical framework

The nature and construction of mathematical objects is a main topic in Mathematics Education (Harel & Tall, 1991; Sfard, 1994, 1998). A fascinating point concerns the "question of the primary sources of our understanding" (Sfard, ibid., p.45), of the genesis of mathematical concepts and of their relationships with pupils' pre- and extra-school experience. A complete investigation of the problem is beyond the purpose of this paper. What we are interested in is the *embodied cognition* perspective (see: Johnson, 1987; Lakoff & Johnson, 1980; Lakoff & Núñez, 1996; Thurston, 1994) with particular respect to the following issues:

- *metaphors* as grounding and producing basic mathematics concepts (Lakoff & Johnson, 1980; Sfard, 1994; Radford, 1999, 2000);
- *mind times*, as mental environments where the subjects put and transform the experienced facts (Boero et al., 1996; Guala & Boero, 1999);
- *narratives*, through which subjects make meanings for what they have previously experienced by building up mathematical stories (Love, 1995; Mason & Heal, 1995; Nemirovski, 1996; Scrivener, 1995).

We shall now elaborate the sense that such notions have within our framework.

The issue of *embodiment*, that is "to put the body back into the mind" (Johnson, 1987, p.xvi), assumes in our research a particular flavour, insofar a mathematical proof may seem very far from our bodily experience, and discovering how a typically conceptual activity, as proving is, has its roots in "perceptual, motor-program, emotional, historical, social and linguistic experience" (Johnson, ibid.), reveals very intriguing. Embodiment and proof become particularly intermingled when pupils interact with an *artefact*, e.g. a mechanism, a computer with a dynamic geometry software (Bartolini Bussi, 1993; Mariotti & Bartolini Bussi, 1998, Bartolini Bussi et al., 1999c), or work within *fields of experience* (Boero et al., 1995b), e.g. the field of shadows (Boero et al., 1995c) or the field of natural numbers, conceived as chains built up from counting processes (Gallistel & Gelman, 1992). In such cases language and time reveal crucial, respectively as a tool and as a mental environment, which support the genetic process from perceptions towards SMOs and mathematical proof.

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[3] The deictic function of language (see Radford, 1999, 2000) allows to indicate directly in the discourse some object which has not a name: words like "this", "that" are typically exploiting a deictic function.

In fact, proving[4] is an activity where discursive and semiotic processes are essential, while the formalistic aspects are not so important (see Arzarello, 2000):

> "to expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim ... that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge" (Lolli, 1999, quoted in Arzarello, 2000).

At this point, mathematical embodiment knocks on the door. But how does this effectively enter the game? We shall argue that language and time are the two right ingredients to look at in order to grasp the genetic processes of pupils towards proof. The approach we shall develop takes into account the semiotic analysis of generalisation processes given by Radford (Radford, 1999, 2000). In a detailed study of novices' performances who, working in group and interacting each other, try to generalise and write in algebraic form regularities that are discovered in so called 'figural numbers', Radford points out that the transition to the abstract general algebraic formula is trigged and supported by two main functions of language, deeply intermingled with the metaphoric function:

(i) the *deictic* function (see note 3);

(ii) the *generative action function* (which supplies the conceptual dimension for generalising: see also the notion of *grounding metaphor for functions* in Lakoff & Núñez, 1996).

According to Radford's analysis, the two functions start and support the genesis of SMOs in algebra (in our terminology): language produces surrogates for (not yet existing) mathematical objects, which are grounded in the subjects' knowledge and fields of experience; metaphors are the tools by which subjects express this link and start creating that conceptual dimension, which will reveal essential for the construction of the mathematical object self.

Our claim is that deictic and generative action functions are present and important also in the geometric context, for example when pupils explore situations and formulate conjectures using Cabri or Mathematical Mechanisms. The way things happen is specific of the artefact. In particular, the generative action of (some types of) dragging reveals crucial (Arzarello et al., 1998b) with Cabri.

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[4] We use the word *proof* referring both to proof as a final (usually written) *product* and to the proving *process* (see Douek, 1999); the meaning we refer to each time will be clear from the context.

Moreover, observations of interaction and of dragging in peers working at the same computer with only one mouse show that a major component of pupils' behaviours and dynamics is time. This component appears also in other contexts: research carried out by Boero and his co-workers (Boero et al. 1996; Guala & Boero, 1999) has shown the relevance of mental times to analyse the components of mental dynamics in pupils working in different contexts. Guala & Boero introduce some categories for mental time: t. of past experience, contemporaneity t., exploration t., synchronous connection t.. All of them are more or less guided by the image of an *order* in a continuum of events (or in more continua), which can be transformed or seen in different ways by the subject. Guala & Boero's categories for mental time are important also in our study, especially for analysing the processes of generation of conditionality.

However, our research has shown also other aspects after which mental time enters into the genesis of SMOs, namely *'tempos'*[5], with consequent problems of *synchronisation*: see 2.2 for examples and comments. 'Tempos' and orders are crucial for the genetic process of SMOs and it will be shown that an artefact like Cabri seems to facilitate their activation [6].

2. Language and Time

Language is a crucial tool through which pupils, possibly with the support of teachers, elaborate their *daily experience* towards more sophisticated behaviours: from expressions describing everyday life (narratives: see below) to sentences which organise the experienced relationships into *causal, final, hypothetical* moves. Because of the verbal coaching, in these processes students' mental times are structured according to a double polarity: *the past*, that is the lived experiences recalled by memory; and *the future*, namely the space of anticipation and volition. In this sense, language and time are essential ingredients in the genesis of *theoretical knowledge*, that is "a system of scientific concepts" in the sense of Vygotsky (1934, chap. VI); such a knowledge is *a-timed*, hence essential transformations are necessary to construct it from experiences which are embedded in time. The evolution of pupils towards it requires a long period of apprenticeship in school and long-term interventions of teachers; developing a theoretical model for describing such an evolution is one of the goals of our research but at the moment it is not yet completely elaborated. Specifically, we are studying the evolution of pupils' *solving abilities* in different contexts and the conditions which seem crucial for causing and supporting such an evolution, including the consequent didactical engineering.

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[5] Other possible words to use are *rhythms* (which have a more external connotation) or *moves* (as actions in a sequence, with different quickness). For an elaboration on such concepts, see the vol. 879 of the *Annals of the New York Academy of Sciences*, 'Tempos in Science and Nature: Structures, Relations, and Complexity', in particular Varela (1999).

[6] It is interesting to observe that 'tempos' are important also in number contexts: in fact they often are the root of fields of experiences for pupils, who acquire the notion of number also through processes of counting down and up; pupils and teachers of elementary schools use the metaphor of the *chain* to describe them (Boero, 1995a). For another example see Bartolini Bussi et al. (1999).

For the clarity of exposition and due to space constraints, we shall limit to illustrate a condensed segment of such an evolution, namely we shall show the genesis of SMO's in students who have already undergone the process of apprenticeship towards theoretical knowledge but who pass again through its main steps in the emblematic example which we shall comment.

We shall start our illustration discussing with more details the analysis of language and time as tools for interpreting the genetic process of SMOs.

2.1 Analysis of Language

A first point to stress is the mixed production of non verbal and oral language (e.g. gestures of pupils who draw geometric figures in the air or on the screen, together with their oral comments): it possibly creates and always supports the partial, generally disconnected, order which is given to the facts experienced by the subjects within their mental time. But an order can exist since the subject can make sense of her/his experiences. At this point *narratives* enter the scene, in different forms. Nemirovsky (1996) defined a mathematical narrative[7] as a narrative articulated with mathematical symbols. This type of narratives is produced for ex. by pupils who reconstruct their past experience with Cabri or the Mathematical Mechanisms while they are moving to the proving phase (see Douek, 1999, for examples in different contexts). But there is also another type of narratives in mathematics, that has been studied by Mason & Heal (1995) and Scrivener (1995), namely the *sketches* (drawn on paper, but also made from gestures in the air). In fact the sketch may be a tool which activates the narrative function, insofar it is made by pure imagination (Scrivener, 1995). Goldschmidt (1991) distinguishes two modalities of visual reasoning while working with sketches: "seeing that" and "seeing as"; there is a dialectic between the two: "a back and forth swaying movement which helps translate particulars of form into generic qualities, and generic rules into specific appearances" (quoted in Scrivener). Hence its functions enter the dialectic between perceptual and conceptual, emphasized by many scholars (Laborde, 1999), and is deeply intermingled with verbal competencies and mental times.

A second issue concerns the deep link between language and pupils' mental dynamics. Mental dynamics 'give life' to perceptions, insofar they can be used to activate the past experience as well as to anticipate a future intention. In such cases, argumentative activity, especially in the interaction among mates (see section 3), has a double function: (i) scaffolding the construction of new (inter- and intra-conceptual) relationships; (ii) controlling and managing the whole (conjecturing, proving, etc.) process. The two functions may be integrated particularly when the discovered relationships, conjectures and proofs allow students to gain new insight in a mathematical problem or field. The link between language and pupils' mental dynamics is particularly interesting within the Cabri environment: very often, the *perception* of objects drawn in Cabri leads students to use *metaphors* and *deictics* to name them; immediately they start actions on them, which modify both the perceptual and the metaphoric aspects (*transformational* function: see Simon, 1996). The interactions among perceptions, metaphors, actions are the starting point of an *evolution* towards the structured mathematical objects.

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[7] Ricoeur, who is often quoted by Nemirovsky, points out that one of the narrative's major functions consists in ordering different times: "the activity of narrating does not consist simply in adding episodes to one another; it also constructs meaningful totalities out of scattered events. The art of narrating...requires that we are able to extract a configuration from a succession" (quoted in Nemirovsky, 1996, p.198).

2.2 Analysis of mental timed frames for 'tempos' and for orders

The evolution from perceptions towards SMOs generally happens within rich *timed frames*, which are ruled by the language and the actions of the subjects. A first frame is nurtured by different **'tempos'** which can be observed, e.g. working with Cabri. A first 'tempo' (present also in paper and pencil environments) consists of the periodic change from *ascending* to *descending control* of the subject with respect to the geometrical figures and backwards (see Saada-Robert, 1989; Arzarello, 1998a): it varies during the performance and marks also the change in the way students see the mathematical objects, with respect to what is considered as given and what is to be found (see section 3). It has low frequencies (in the first 20 minutes of the video we have counted about 15 such changes): for this reason, it is called a *slow 'tempo'*.

Another kind of 'tempos', which are faster, are present in the interaction between pupils and the software Cabri. Their roots seem to consist in the rupture of the synchronism which exists between thought and physical movements of our body (see the description in Berthoz, 1997, and Varela, 1999), when subjects use the mouse in a dynamic geometry software. The phenomenon is typical of novices but is present also in subjects who have more experience with the software (#61-64, #128)[8]; an example is given by group interactions, when the mouse is in the hand of one subject and the others cannot follow what happens on the screen (#92 and before). Generally the subject tries to overcome the gap between the two 'tempos', controlling hand gestures and dragging slowly and carefully so that the synchronism between perceptual and cognitive aspects can be established (#77, #102). Such 'tempos' are faster (Berthoz, 1997; Varela, 1999), hence they are called *fast 'tempos'*.

'Tempos' may support a continuity and correspondence between perception and thought, back and forth from inside to outside (see the genesis of circumcentre in the Conclusion). It seems that a feeble co-ordination of fast 'tempos' is an obstacle for solving problems using dynamic geometric software. On the contrary, the students who master Cabri have a good co-ordination of fast 'tempos' in that context and this has positive consequences at the cognitive level also for the control of slow 'tempos', hence for the whole process of problem solving in Cabri. Moreover, it seems that some functions of the dynamic software (specifically, some types of dragging) are useful for generating a strong evolution from perceptions towards SMOs. In this sense, a conscious use of dragging (which requires a high co-ordination of fast 'tempos') can support the subject in the processes of generating generalities. This has radical consequences for the teaching: promoting a conscious use of dragging can be achieved only by suitable interventions of the teacher, who scaffolds the experiences of pupils and teach them the different typologies of dragging (the method is systematically used in the classrooms of our project).

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[8] #n indicates the sentence numbered n in the Appendix.

In the case of numbers, the 'tempos' of counting can support the natural order of numbers and become an essential constituent of their field of experience: the search for a regularity starts with metaphors which try to build a new order from the facts experienced through numerical explorations within such a field and evolve towards suitable SMOs (see Boero et al., 1995a).

Language encompasses also the problems of the **order** of events, which appear not linearly ordered in perceptual experiences (Varela, 1999). Here the narrative function of language plays an essential role: it allows students to make sense of what has happened, connecting the perceptual experiences with the past and categorised ones. In other words, while metaphors and deictics allow the construction of objects in a discursive form and the beginning of a generalisation process which anticipates the *future*, the narrative function allows events to be ordered by looking mainly at the *past* (see: #104, #138 , #151, #152). The result of the evolution may be a *de-timed* sentence expressed in the *present tense*[9], which has the features of a scientific discourse.

Mental time becomes a sort of mental environment where subjects use language to order in some way the facts that they have experienced (see Varela, 1999). For ex., gestures and the broken oral language in #73 are an attempt to grasp the complex relationships among geometrical objects represented by Cabri figures; that is, subjects try to express discursively a non linearly ordered factual situation (i.e., Cabri figures which change because of dragging) with a tool (oral language) which is more comfortable with linear order[10]. This creates a tension which is expressed in the example by the broken sentences and utterances, as if language tried to mimic the complex order of the experienced facts.

An interesting point to stress is the fact that the variety of languages used by students in most of their performances helps communication. Illuminating examples are given by # 65, 66, 67 and # 73, 74: the verbal part of the message apparently has no meaning; on the contrary, the pupils successfully communicate some essential point to their mates. The non-linear structure of her sentences reflects the different nature of what she is communicating: this can happen when all the 'tempos' have been synchronised among the girls and the perceptual and metaphorical ground of what has happened on the screen is shared among the three pupils. Consequently, the non linear order constitutes a common basis upon which they can share some ideas about the SMOs they are building: we call *multivariate* such a cluster of languages used together. The timed frame for orders constitutes a ground for the *cognitive unity* discussed by Garuti et al. (1996, 1998): in fact students can use a multivariate language to describe (#78) and communicate (#73) their ideas. Only after this multivariate language has done its job, namely has guaranteed a shared experience for the three girls, they start a further evolution process (namely from #127 to #152) towards a linear and more formal language, which explains in more canonical forms why what they have experienced is such (#150). The evolution from Perceptions-Deictics-Actions towards Mathematical Objects-Symbols-Relationships (that is SMOs) described till now can be sketched like in fig.1.

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[9] The language we refer to is Italian: the sense of its present tense is possibly different from English.

[10] See Simone, 2000, for a discussion on linear and non-linear languages.

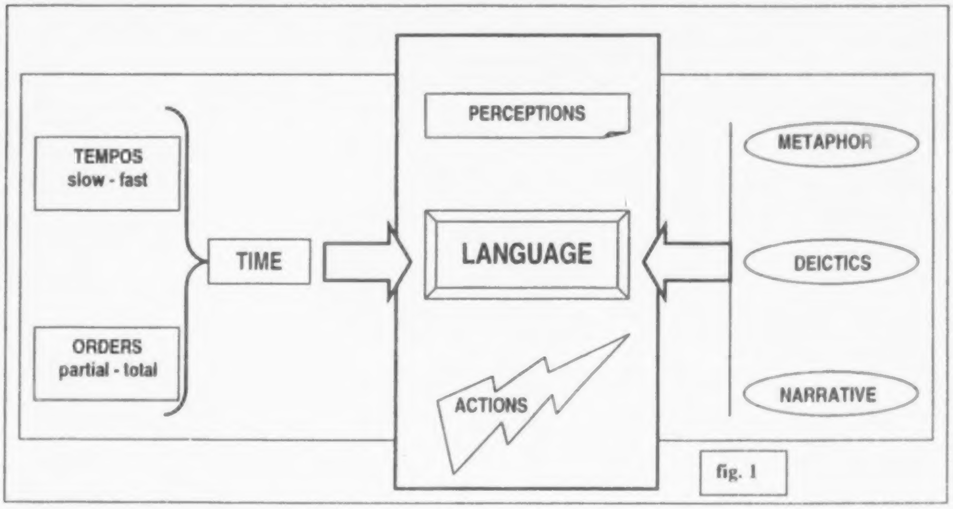

fig. 1

2.3 Other theoretical tools

Even if time and language are the two major ingredients of the model, the analysis of SMOs requires also many other tools; more precisely:

- the theory of *experience fields* (Boero et al., 1995b);
- the *discussion as a polyphony of voices* (Bartolini Bussi, 1996), typically when one voice is that of the official scientific knowledge and the others those of pupils;
- the notion of *semiotic mediation*, from the Vygotskyan historical-cultural school of psychology (Wertsch, 1998; Bartolini Bussi & Mariotti, 1999a, 1999b);
- *typologies of dragging* analysed by Arzarello and Olivero (Arzarello et al., 1998b).

Moreover, a new tool is in course of elaboration, namely the analysis of the *neurological bases* of some mathematical concepts (Berthoz, 1997; Dehaene et al., 1993, 2000; Longo, 1997): particularly the analysis of *perception* as a multisensorial integration, and of *action* as anticipation of movement given by Berthoz (1997) and discussed by Longo (1997) from the mathematical point of view.

Space constraints do not even allow us to sketch the issues mentioned above: the interested reader is invited to read the quoted reference for more information. The next section will use all the ingredients previously outlined to describe the genesis of SMOs in a concrete emblematic example.

3. A case analysis

Following up these general comments, we shall now give a more detailed analysis of the protocol in Appendix. In the sequel, H will indicate the dynamic figure ABCD and T the figure A'B'C'D'; in the protocol, the way pupils look at the two figures changes over time: sometimes they look at T as depending on H (we write $H \Rightarrow T$), sometimes students look for hypotheses on H so that T satisfies particular conditions (we shall write $T \Leftarrow H$). The different directions through which pupils look at the objects are deeply intermingled with their mental times, e.g.: order from past to present or from future to present, etc.; 'tempos' depending on the control with respect to the objects (ascending vs. descending), to the dialectic with dragging and sketches and to the outside-inside dynamics and back. 'Tempos', orders, causal and conditional dependencies find here a crucial connection: in particular the different orders that subjects attach to experienced facts may change or not in the transition from exploring to conjecturing and to proving and this may have consequences on the degree of difficulties found by pupils (see also Douek, 1999).

The starting point of the cognitive dynamic is #37, where M makes explicit the heuristics to be used, which is developed in the following items (till #48): this is the genesis of $H \Rightarrow T$, which lasts till #60. In #50 the degenerate point, which will be named in #58 begins to be present as a perceptual fact. The episode marks the beginning of a systematic exploration (descending control and attention towards the future). During the exploration (#52) the theory crops up from the past: together with the perceived object it produces the sentence #53, which is spelled in the de-timed present tense of standard scientific sentences. In #58-60 the genesis of SMOs starts: the point (of intersection of the perpendicular bisectors, when it exists) is manipulated through dragging and its meaning is framed into conditional sentences, that is the perceptual facts and the generalising function interact to generate a mathematical sense for what is perceived (#60). This produces an inversion in the way the objects are looked at ($T \Leftarrow H$), which culminates in #64; similarly the control becomes ascending, as it is shown by the *wandering dragging* (we use the analysis of dragging developed by Arzarello et al., 1998b) pursued in #61-64. There are also different 'tempos' (#61-67) in the students: E explores with Cabri (#62, 64: *wandering dragging*); V wants to think with her own 'tempo', which is different from E's (#65):

in fact she does not grasp the explanation of E (#60); M tries to synchronise the two (#66). In #68 E draws a parallelogram, which marks a new inversion (H ⇒ T, descending control). The tension between perceptual and theoretic aspects is always high (#72): in #73 the multivariate language, which collects perceptual and theoretical chunks, allows V to synchronise her 'tempo' with that of E and to communicate with M (#74); namely, multi-variety and non-linearity of language allow communication and sharing of knowledge among the students (see the comments in 2.2). In fact, in #75 E transforms the sentence of V into a "linear" sentence, structured in more a canonical way, and in #76 V echoes the voice of E. In #77 there is a new inversion (T ⇐ H, ascending control): the 'tempo' of E has changed and it seems it is now shared in the group. The slow 'tempos' of her gestures and the new modality of dragging (the so called *bounded dragging*) allow E and her mates to rule the guided exploration, which now starts from an already structured object. In this sense, the action with the mouse and the consequent dragging, with its complex typologies, can incorporate a deictic and a generalising function.

In #78-84 there is a systematic genesis of structured objects (different types of quadrilaterals): also Boero et al. (1996) observed this type of exploration in other contexts. The cases of the square and the rectangle are object of a mental experiment: E makes a mental dynamic exploration, namely she activates a strategy which is similar (but not identical) to the "changing hypothesis" strategy described by Boero (ibid.). The mental experiment produces a deduction in the form of impossible examples. In #82-91 exploration continues: there are many inversions of control and of the relationships between H and T; #92 is interesting: E makes a mental exploration supported by her hands' gestures (T ⇐ H), which is transformed into "linear" language in #94. After the new 'impossible example' of #95 (ascending control), the degeneration is grasped at a new (more theoretical) level, which is marked by the linguistic transformations of #96, #100, #101, #102, #103. The last sentence expresses an *abduction* (Arzarello et al., 1998a), which is produced after the *lieu muet dragging* (Arzarello et al., 1998b) in #102, where B describes a circle. The linguistic transformations mark the abduction while the effects of dragging suggest the theory of remarkable points of a triangle, insofar they are linked to inscribed and circumscribed circles. Such a theory is evoked by E and the time is now towards the past (#104). In #104-126 the students give voice to the theories and discuss them; the resolving point is #126, where there is the genesis of the right figure. In #127 E gives *the* explanation in a multivariate sentence, in which however the theoretical side evoked in previous interventions (#104-125) is marked by the present (de-timed) tense. It is interesting to observe that in this present time do live both words and deictics but there is no reference to the particular figure on the screen in that moment: in fact the argument is supported by a sketch made on the screen by hands, which refers to a *general* geometrical object (Harel & Tall, 1991). Namely, because of the narrative incorporated in the sketch, thinking can go beyond the concrete perceptual aspects and get the theoretical side (they do not see "that" but "as", see 2.1). In fact, in #131 the "multivariate" sentence of #127 is transformed into an almost linear utterance, which incorporates also some chunks of the logical relationships among its constituent sentences.

Conclusions

The comment above illustrates how embodied is the generation of SMOs and how strong the cognitive unity from the first empirical perceptions to the production of logically structured sentences of mathematics can be. This is achieved insofar the students have interiorised the dragging practice through a cognitive apprenticeship (Arzarello et al., 1993), where the role of teacher is basic.

The written proof produced later on by E, V and M exhibits a strong continuity through linguistic transformations and (modulo some inversions between H and T) from all the explorations above to the final product, which has the canonical structure of usual mathematical proofs (something similar is described in Boero et al., 1996). Constraints of space do not allow us to describe it here. We limit ourselves to sketch the structure of the whole process in fig.2, where the main transformations are shown, namely: the genesis of conditional statements through *de-timing*; that of mathematical sentences through *linearisation* and that of abstract concepts (and their symbolic representation) as a *generalising transformation* of metaphors, deictics and narratives.

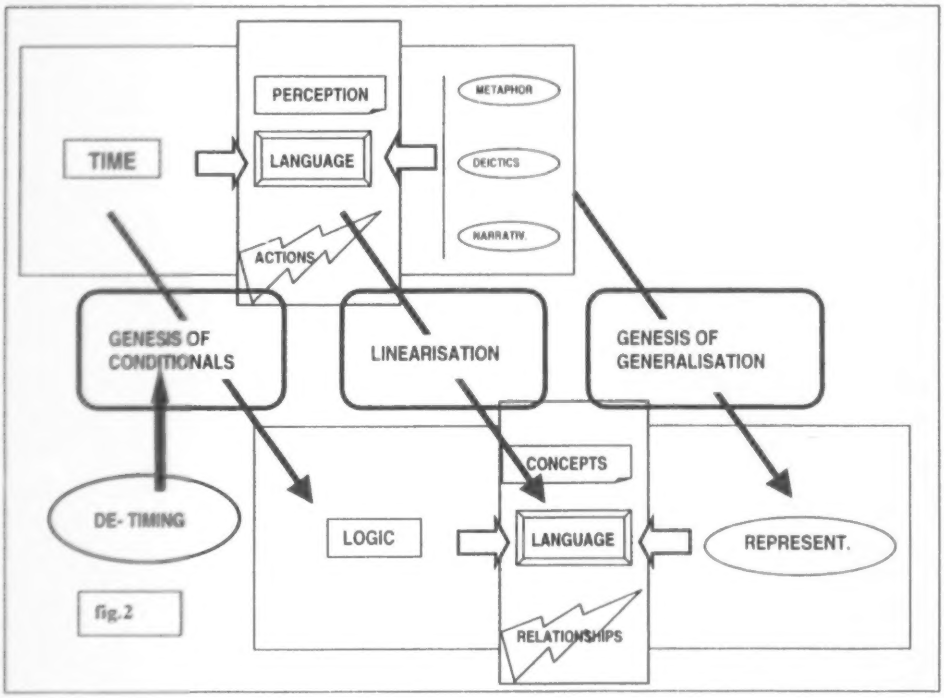
PERCEPTION
TIME
LANGUAGE
ACTIONS
METAPHOR
DEICTICS
NARRATIV.
GENESIS OF CONDITIONALS
LINEARISATION
GENESIS OF GENERALISATION
CONCEPTS
DE- TIMING
LOGIC
LANGUAGE
REPRESENT.
RELATIONSHIPS

fig.2

Now, let us summarise the main points. We have illustrated the generation of SMOs in geometry within the embodied cognition perspective, analysing processes of pupils working with Cabri. Mental times of students revealed crucial in managing such a generation. Another crucial point was the mediation of the artefact, in particular the role of dragging and of the representations of mathematical objects (like Cabri drawings, sketches); the dynamic geometric figures have proved a *cognitive pivot*: the hypotheses incorporated in them are transformed into new ones through the complex dynamic of action and interpretation within changing tempos and orders of the subjects when interacting. An emblematic example is the story of the "degenerate point". At the beginning (#50) it is a purely perceptual fact; then it is a metaphor which describes the result of a dynamic process (#58), but whose meaning remains to be explained (#65); the first chunk of theoretical explanation is obtained (#72) through the framing of the perceptual experience in a narrative which encompasses different perceptions (#66 and ff.). In the end (#96) the word "degenerate" marks the conclusion of a rich exploration framed within a time section in the sense of Boero et al. (1996). Now things are ready for the final transformation, namely the "degenerate point" to be transformed in the "circumcentre" (#102, #127), whose genesis is the dragging described in #100 through the metaphor of "keeping the point inside".

Appendix: an emblematic protocol.

PROBLEM. *You are given a quadrilateral* ABCD. *Construct the perpendicular bisectors of its sides:* a *of* AB, b *of* BC, c *of* CD, d *of* DA. A' *is the intersection point of* a *and* b, B' *of* b *and* c, C' *of* c *and* d, D' *of* a *and* d. *Investigate how* A'B'C'D' *changes in relation to* ABCD. *Prove your conjectures.* [17-18 year old pupils, who know CabriI very well and are acquainted to explore situations when presented with open problems). We present some excerpts of the protocol of Elisa, Michela and Valentina's work with Cabri; the protocol is the transcription of a video, which lasts one and a half hour. We present some (parts of) episodes of the first part (which lasts 20 minutes altogether), where the genesis of SMOs is particularly evident. The three students work with one computer and have also paper and pencil at their disposal].

The first 4 minutes of the video, with 35 interventions of pupils, are skipped: the students read the problem, draw the figures, accomplish the constructions and give names to the created objects.

The students (E, M, V) constructed fig. 1 and checked its correctness through dragging.

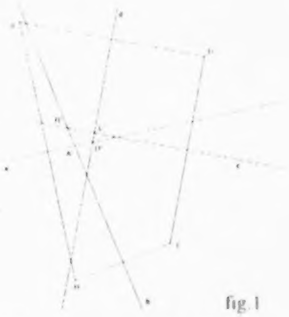

fig.1

36. E: and now? (E has the mouse)

37. M: One must see how it varies, as the external quadrilateral changes [ABCD]

...

41. V: I think that not...try moving...the figure...[E drags randomly point D] ... 'cause.... move this one [V indicates point B and E drags it randomly]...it seemed to me that you had put the...you know...the function of the segment, that you can create without doing the points...it seemed that you had not shot this one [A']...do you understand?

.....

(time = 5' 25'')

48. V:...but if you already do it coloured...

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you get a small coloured point.
[E colours the quadrilateral A'B'C'D' and drags the point D]
49. E. And let's try perhaps...let's try to see what happens with regular external quadrilaterals...
50 M: I don't know...let's start with a square, so that we see...
[E drags B, C, D up to get a rectangle]
51. V: Properties of the perpendicular bisectors? [the Italian word is *assi*]
52. E: The perpendicular bisector... how was it?
53. M: Hence...the perpendicular bisector passes through the midpoint...
54. E: It is perpendicular!...
55. M: ok!
56. E: Well in the square, in the...
[She measures the sides of ABCD, then drags randomly first point C then point B]
(time = 7' 10")
57. E: No, I was wondering...that...I was wondering!
[E stops the dragging with fig. 2]

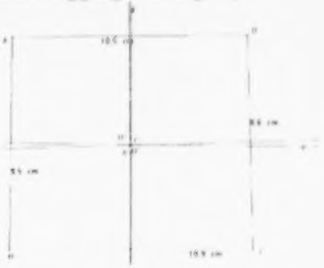

fig.2

58. E: No, that is...it degenerates into a point...it's logical isn't it?...if they are parallel...that is, if the sides are perpendicular...[she drags B]...
59. M: ...we are looking for...
60. E: I mean if the opposite sides are parallel [she continues dragging B], those [the perp. bisectors] are perpendicular. And up to this...Isn't it?....and if they are equal the midpoint is on the same line. [She drags C till ABCD becomes a rectangle].
61. M: ok...so?
[E drags A randomly, then D; in the end she goes back to the original figure]
(time = 9')
62. E: Please, tell me something!
[E drags B, D, C, A systematically]
63. V: What are you doing? Are you moving randomly?
64. E: No, I was wondering if I could construct a figure...
65. V: Listen to me, please; let's try thinking... just a moment...'cause of that we have done before...to finish the discourse, when it degenerates into a point, that...have I misunderstood or we have not explained it?
66. M: well, practically she is saying: since the properties of the perpendicular bisector are perpendicularity and the distance from a point...if...the different segments are parallel, then since they are perpendicular....Moreover if two of...like in a square for example, the midpoint must belong to the same straight line.
67. V: yes
[In the meanwhile E has dragged the points A, B, C, D in order to get a parallelogram]
68. E: I am doing a parallelogram...the sides are parallel, aren't they? in the parallelogram. Hence also the perpendicular bisectors are parallel, isn't it? They are parallel two by two.
69. V&M: yes
70. E: So also the segments A'B' and C'D' are parallel.
71. V: Hence it maintains...no, nothing!
(time = 11')
[E drags the points B, C, D till she gets a rectangle]
72. E: hence the square has been proved...degenerate...
73. V: Hence if...when...and hmm, yes, that is natural, because when there are two...the two sides of the external one...the two sides parallel two by two, it is natural...that is it should always be that the perpendicular bisectors are...
74. M: it is so.
75. E: Because they are parallel...they are perpendicular to two parallel lines.
76. V:...they are parallel...
77. E: let's move the point very slowly to see what changes [she drags the point C for a while]. Now they are not any longer parallel, hence...these two [d, b] are not any longer parallel...sure, it is logic...and not for these two [a, c] ...That is what we have said up now.
[E drags slowly point C along line BC and back]

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78. E: Nothing, there is no way to get...the...that they are parallel...a square... type: inside there is a square; the sides should be parallel two by two...However one longer than the other! Isn't it? ...But it cannot ever be, because otherwise...then these [the sides of ABCD] are not any longer parallel.
79. M: It cannot be a square inside!
80. E: Sure!
81. M: Neither a rectangle!
82. E: It can be only...a trapezium [she drags C trying to get a trapezium]
83. M: Or a parallelogram.
84. E: A parallelogram or a trapezium.
85. V: I am wondering why...
...*(time = 14')*
[M takes the mouse and drags B]
92. E: No! Why the lines...should be so [she mimics with her hands two parallel horizontal lines]; then it means that one is longer than the other, isn't it?
93. V: sure! necessarily!
94. E: However if one is longer than the other, the other two are not parallel any longer...Otherwise in the parallelogram they have moved , since they all are parallel.
[M drags B and stops when the perpendicular bisectors coincide in one point].
95. V: Hence it can never be..
96. E: My God! It degenerates into a point again!
97. E: Excuse me, can I...? [she asks to move the mouse again and drags C].
98: Let us see what remains equal, when it remains...
99: V&M: when?
100. E: That is, I try dragging the stuff....and...the point...keeping the point inside...that is moving the point, but leaving that...that...the quadrilateral degenerates into a point.
101. M: That is, it keeps inside...
(time = 16' 02")
102. E: ...to find a property that...when it degenerates into a point...do you understand? [she drags the point B slowly, trying to keep the perpendicular bisectors coincide]
103. E: Let's mark the angles...excuse me, but in a triangle the intersection point of the perpendicular bisectors is...
104. V: hmm, I had already thought of that!
...[the students recall what they remember about the remarkable points of a triangle; they are in doubt whether to consider the circumcentre or the centroid; they make some exploration and discuss about the circles outside and inside the quadrilateral]...
126. M: That is when you can put a circle inside.
127. E: No, no! I know it! It is the circumcentre..., why it must be equidistant from the sides, isn't it? [she indicates with fingers on the screen] This point [the meeting point of the bisectors] is the perpendicular bisector of this [AD] hence it is equidistant [from A and D].
(time = 18' 13")
128. V: Wait a moment, stop please!
129. Hence this point here [the supposed circumcentre]...if you have...look at..., that is the bisector, it is equidistant from the extremes...
130. V: sure!
131. E: ...because the bisector is the locus of points which are equidistant from the extremes...hence it is equidistant from this and from this [A and B]. But from this [A], these two are equal...[she repeats the same reasoning and gestures with respect to all vertices]...hence in the end they are all equal and it is the ray, isn't it?...
132. V&M: yes!
133. E: ...I mean I wasn't that clear.
134. V: Try to do a circle.
135. M: Try!
136. E: hmm...how can I do it?
137. M: excuse me...centre and point; make the centre here [the intersection point of bisectors]...
138. E: Wait! No! It is enough to see...the angles. How are they when...it is a cyclic quadrilateral...the opposite...are 180°? [they ask the teacher the right property and then measure the angles in Cabri; E measures angles D and B; some explorations and reasoning concerning the sum of opposite angles D and B]
...*(time = 20' 43")*
150. E:....yes, in conclusion....hmm, anyway the proof is that...the perpendicular bisectors are equidistant from vertices...hence this point [the intersection point of the bisectors] is

equidistant from these two vertices [A, B] and also from these [A, D], from these [D,C] and from these [C,B]...isn't it?...then in the end ...Isn't similar to a problem we have already done?

151. V: We certainly did something similar!

152. E: perhaps with the medians.

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Teaching by Open-Approach Method in Japanese Mathematics Classroom

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ORIGINS OF OPEN-APPROACH METHOD IN JAPAN

In Japan, mathematics educators have traditionally been emphasizing mathematical perspectives in their research and practice. In these twenty years, more attention has been paid to individual students in the stream of mathematical perspectives emphasized. Some of the representative research results have been published under the titles of "The open-ended approach," "The open approach," "From problem to problem" and "Various ways of thinking" (Shimada, 1977; Nohda, 1983; Takeuchi & Sawada, 1984; Sawada & Sakai, 1995; Koto, 1992). The tradition of posing and solving problems in mathematics class since before World War II served as a base for the emergence of these researches.

Most of these recent researches focus on possibilities of individual students as well as their mathematical ways of thinking. Development of teaching methods that are tuned to a variety of students' ways of thinking is also a major issue. In other words, students' mathematical thinking, mathematical perspectives and development of teaching methods have been integrated, which constitutes a remarkable feature in recent Japanese mathematics education and mathematics education research.

An origin of such trend was the research on evaluation conducted at the beginning of 1970s. One of leading research was the one by Shimada and others concerning the method of evaluating student's achievement in higher objectives of mathematics education. They meant higher objectives as follows:

- To be able to mathematize a situation and to deal with it. (In other words, to be able to bring forth an (important) aspect of the problem into student's favored way of thinking by mobilizing their repertories of learned mathematics, to reinterpret it in order to deal with the situation mathematically, and then to apply their preferred techniques.) (Shimada, 1977)
- To be able to collaborate with others in solving a mathematical problem. (Shimada et al., 1972)

It was here that they developed open-ended problems in order to evaluate students' activity. Open-ended problem refers to the problem that is formulated so as to have multiple correct answers. Shimada and others developed different open-ended problems such as "marble problem" and "water flask problem."

In those days, Japanese Mathematics Course of Study was organized around the idea of Modernization of Mathematics Education. The class activity was so called "issei-jugyou" (frontal teaching). There were 45 students and one teacher in the classroom. The teacher explained new concept to the students and presented examples of concept and solutions of exemplary problems. A series of knowledge, skills, concepts, principles and laws was presented to students in the step-by-step fashion. Under such circumstances, the open-ended problem was expected to serve as a vehicle for changing the lesson organization substantially.

In the beginning, the research was conducted by four researchers, Shigeru Shimada, Toshio Sawada, Yoshihiko Hashimoto, and Kenichi Shibuya. A few years' later, more researchers and teachers in elementary and secondary schools participated in the research. These teachers used the method in their mathematics classrooms. The book "The open-ended approach: A new proposal for teaching mathematics" (Shimada, 1977) was published as the result of this collaborative work. Recently, the book was translated into English and published by NCTM (Becker & Shimada, 1997). The research has been continued and developed in the ways as mentioned above.

At present, there are still many schools in Japan that have 40 students and a teacher in one classroom. However, ways of teaching became more variable compared with 30 years' ago, and came to emphasize ideas of each student together with the traditional mathematical perspectives. As indicated by the production of many books above, the idea of "openness" in teaching and evaluation has been developed and extended in various ways through collaboration between researchers and schoolteachers and has been realized in actual mathematics classrooms in Japan (see Note).

In this short presentation, I will describe the idea of open-approach method and show an example of teaching situations that realized the idea of open-approach method. Then, I will discuss several perspectives for future mathematics education research from the viewpoint of open-approach method.

IDEAS OF OPEN-APPROACH METHOD

Opening Up the Hearts of Students Toward Mathematics

All of the educational activities should open a student's present-day learning to his/her future learning. Thereby, the student can acquire necessary qualifications to make his/her life successful. Even in school mathematics, we should take into account that every student is encouraged to seek his/her own way of life, and has whole mind and body to contribute to his/her community with full force on the basis of mathematical sense, knowledge, skills, the ways of thinking, and so on. Therefore, we should ensure the maximum opportunity and the best environment of learning in any kinds of educational activities as possible. However, it is clear that most students cannot necessarily learn the content of more than the middle grades, because of the “hard” characteristics of school mathematics (its structural, abstract, and conventional phases), even though they can easily learn the content of the lower grades by themselves (Nohda, 1982). Therefore, the appropriate teaching is necessary especially in school mathematics.

In teaching mathematics, teachers are supposed to assist their students in understanding and elaborating their mathematical ideas as far as possible in response to students' achievement, disposition, and so on. However, the teaching only anchored in the logic of teacher never can open up the heart of student, even if its process and product are “attractive” for teachers mathematically. On the other hand, the teaching flattered students' ideas is bound to end up the activities of low mathematical quality, and finally never can open toward mathematics.

Teaching by open-approach method aims that all students can learn mathematics in response to their own mathematical power, accompanying with certain degree of self-determination of their learning, and can elaborate the quality of their process and products toward mathematics. In other words, teachers who employ open-approach method in their teaching need to try to understand a lot of students' ideas as possible, to sophisticate the ideas in mathematical activities by means of students' negotiations with others and/or teachers' advice, and to encourage their self-government in elaborating the activity mathematically. Thus, the teaching by open-approach method intends to open up the hearts of students toward mathematics.

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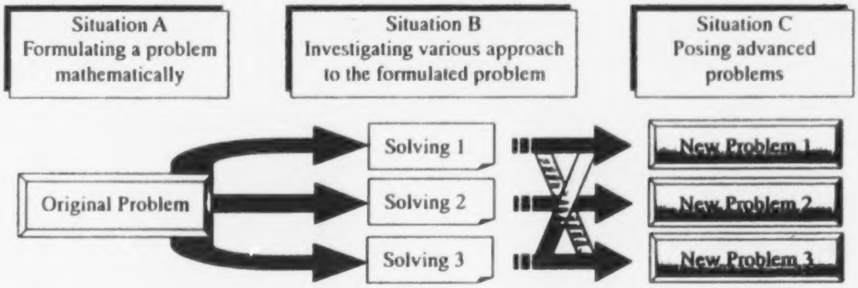
The teaching by open-approach method assumes three principles. The first is related to the autonomy of students' activities. It requires that we should appreciate the value of students' activities for fear of being just non-interfering. The second is related to the evolutionary and integral nature of mathematical knowledge. Content of mathematics is theoretical and systematic. Therefore, the more essential certain knowledge is, the more comprehensively it derives analogical, special and general knowledge. Metaphorically, more essential knowledge opens the door ahead more widely. At the same time, the essential original knowledge can be reflected on many times later in the course of evolution of mathematical knowledge. This repeated reflection on the original knowledge is a driving force to continue to step forward across the door. The third is related to teachers' expedient decision-making in class. In mathematics class, teachers often encounter students' unexpected ideas. In this bout, teachers have an important role to give the ideas full play, and to take into account that other students can also understand real amount of the unexpected ideas.

Teaching by open-approach method consists of three situations in general; Situation A: Formulating a problem mathematically, Situation B: Investigating various approach to the formulated problem, and Situation C: Posing advanced problems.


Situation A
Formulating a problem mathematically
Situation B
Investigating various approach to the formulated problem
Situation C
Posing advanced problems
Original Problem
Solving 1
Solving 2
Solving 3
New Problem 1
New Problem 2
New Problem 3

In Situation A "Formulating a problem mathematically," teachers show students the original situations or problems, and students try to formulate them as mathematical problems in response to their own learning experience. In Situation B "Investigating various approach to the formulated problem," students are expected to find their own solutions on the basis of their experience. Teachers direct students to discuss the relations of wide variety of solutions proposed, and lead them to integrate seemingly unrelated solutions into a more sophisticated one. In Situation C "Posing advanced problems," students try to pose more general problems on the basis of their activities in Situation B.

Through solving these problems, they are expected to find more general solutions.

Openness and Types of Problems

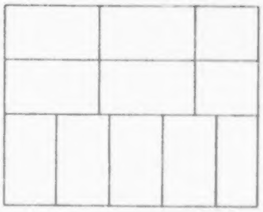
In the open-ended approach proposed by Shimada, emphasis was placed on the problem whose end was not closed to one answer. He and his colleagues intended to organize class by making use of multiple correct answers positively. In the open-approach method, the meaning of openness is considered broadly than the open-ended approach. Here, in addition to the problem whose end was open, the problem that produces multiple correct solutions and the problem that produces multiple problems are included. By this extension, the difficulty of constructing the open problem is overcome. Moreover, it becomes possible to provide more opportunities for students with different abilities and needs to participate in the class. After getting multiple solutions by his/her own, it also becomes possible to lead students to sum up their solutions from the viewpoint of mathematical ideas (Nohda, 1983).

Problems used in the open-approach method are non-routine problems. Furthermore, based on the openness described above, it is reasonable to classify the problems into three types: "Process is open," "End products are open" and "Ways to develop are open." Several researchers use these names. The types are described below with typical examples.

Process is open. This type of problem have multiple correct ways of solving the original problem. Needless to say, all mathematical problems are inherently open in this sense. However, the issue is that many school problems require only the answers or do not emphasize the process aspect of the problems. It is therefore important to verbalize that the process is open and ask for teachers to look at the problems at hand from such a viewpoint. The "card problem" below is one example of this type.

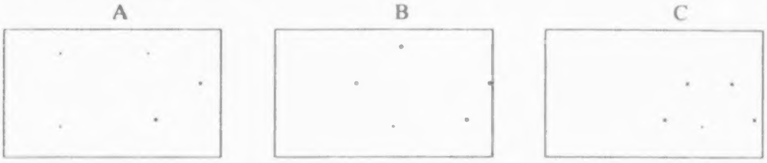
"As 37 pupils will make birthday cards for the teacher 'Matsui Sensei' in the classroom meeting, it has been decided that everyone will make cards. Then, they have to make some small cards (in the shape of a rectangle 15 cm long and 10 cm wide) from some bigger rectangular sheet (45 cm long and 35 cm wide). The problem will be, "How many small cards can you make from the bigger one?"

Here, students may dissect the rectangular sheet into the size of card and get the arrangement as shown in the figure at right. Students also may calculate (35 x 45) ÷ (15 x 10) and get the answer 10.5 numerically. Another student may calculate (7x9) ÷ (3x2) by noting the ratios.



Multiple solutions enable students to carry out the activity according to their abilities and interests, and then through group discussion, to seek a better process of problem solving.

End products are open. This type of problem has multiple correct answers. As stated above, Shimada and his colleagues have been developed this type of problems (e.g., Shimada, 1977). In Europe, Christansen & Walter (1986) talked about the importance of investigation problems, which is similar to the problems that the end products are open. An example of this type, "marble problem," is shown below, which is well known as a representative problem in the open-ended approach.

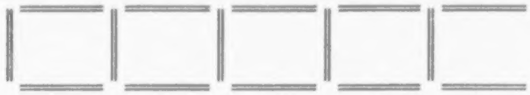


"The figure shows scattering patterns of marbles thrown by three students A, B and C. In this game, the student who has the smallest scatter is the winner. In these examples, the degree of scattering ranges from A to C. In such a case, it is convenient if we have some numerical measure to indicate the degree of scattering. Then, think about it from various points of view, and show ways of indicating the degree of scattering by itemized statements. After that, think of the best answer for this problem."

Students may discover "measure the area of a polygonal figure" as a measure of degree of scattering. Another students may think of "measure of the length of all segments connecting two points," and still another may do "measure the radius of the smallest circle including all points." These methods of measure have advantages and disadvantages. The teacher can help students see both the advantages and disadvantages in generalizing the proposed methods.

Ways to develop are open. After students solved the problem, they can develop new problems by changing the conditions or attributions of the original problem. When we emphasize this aspect of "from problem to problem" (Takeuchi & Sawada, 1984), the problem can be said that ways to develop are open. An example below, "matchstick problem," is taken from problems used in the US-Japan comparative study on mathematical problem solving (Miwa, 1992).

"Squares are made using matchsticks as shown in the picture below. When the number of squares is eight, how many matchsticks are used?



(1) *Write a way of solution and the answer to the problem above.*
(2) *Make up your own problems like the one above and write them down. Make as many problems as you can. You don't need to find the answers to your problems.*
(3) *Choose the one problem you think is best from those you wrote down above, and write the number of the problem in the space. Write down the reason(s) you think it is best."*

Here, students may develop problems by changing the number of squares. They may further change the condition "square" to "triangle" or "diamond," for example. They may also develop problems to ask for the number of squares when the number of matchsticks used is given (inverse problem). Students can enjoy developing their own problems. Furthermore, by comparing with their peers they can discuss mathematical structures of the problem and generalizability of their solutions in the lesson.

Evaluation of Students' Responses

It would be worthwhile to mention here how to evaluate student's activity in the open-approach method. This is because the aim of the method is not to produce correct answers but to promote student's mathematical ways of thinking and creativity. Indeed, it is not easy for the teacher to evaluate a variety of student's responses being produced.

Student's response is evaluated according to the following criteria (see Shimada, 1977)

- Fluency – how many solutions can each student produce?
- Flexibility – how many different mathematical ideas can each student discover?
- Originality – to what degree is student's idea original?
- Elegance – to what degree is student's expression of his or her idea simple and clear?

These criteria need to be evaluated by both quantitatively and qualitatively. Here, especially the first two criteria can be evaluated by counting the number of responses.

In Nohda (1998), a model in the form of a matrix has been constructed to evaluate responses by the criteria of "diversity" and "generality." In the matrix, an item (Aij) shows the number of responses by the student. "Diversity" is expressed by (Aij) where j is a constant, in which different mathematical ideas correspond to different (Aij). "Generality" is expressed by (Aij) where i is a constant, in which different level of generality correspond to different (Aij). For instance, responses of "marble problem" described above can be evaluated in the following way.

Diversity — A1j: Ideas of length, A2j: Ideas of area, A3j: Ideas of variance
Generality — Ai1: Concrete example, Ai2: Semi-concrete example, Ai3: Abstract example

A11: Max. or min. length of two points.
A12: Circumference of 5 points.
A13: Sum of all lengths of 5 points distances.
A21: Min. square covering 5 points.
A22: Min. circle covering 5 points.
A23: Sum of the areas of triangles formed from 5 points and so on.

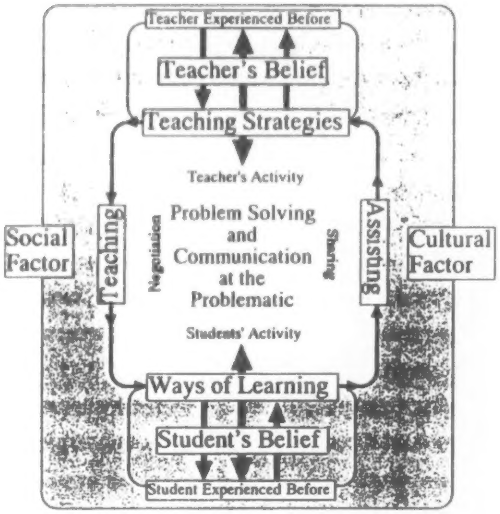
.....

$$P:\begin{bmatrix}110\\000\\000\end{bmatrix}\quad Q:\begin{bmatrix}121\\011\\000\end{bmatrix}$$

According to this evaluation, we can say that the student Q has a more diversified and a more general approach than student P. On the other hand, supposing that P and Q indicate the states of the same student prior to and after the lesson, respectively, then it is possible to know how the student has changed through teaching using the open problems by comparing the two matrices.

TEACHING SITUATION BY OPEN-APPROACH METHOD

In this section, I will describe how mathematics teaching proceeds by using open-approach method in class. Figure below shows a characterization of Japanese teaching of mathematical problem solving (Nohda & Shimizu, 1989). In Nohda (1982), I investigated the process in terms of pedagogical tactics by Herbert. The figure also shows several features of Japanese class in that problem situation that contains important mathematical ideas is presented to students, and students challenge the situation collaboratively and finally reach their solutions (see also TIMSS results by US Department of Education, 1996). However, it becomes more difficult to make such process happen, as students become older and their abilities and beliefs vary far more. Therefore, in the open-approach method, it is intended to provide students with rich situations by using open problems that have possibility to serve for individual differences

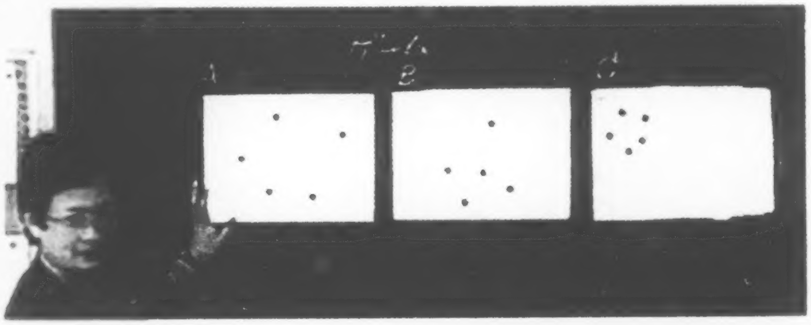

Teacher Experienced Before
Teacher's Belief
Teaching Strategies
Teacher's Activity
Social Factor
Teaching
Negotiation
Problem Solving and Communication at the Problematic
Sharing
Assisting
Cultural Factor
Students' Activity
Ways of Learning
Student's Belief
Student Experienced Before

among students both in their abilities and interests and in the development of mathematical ways of thinking, and to support the investigative process of solving and generating problems. Through such activities, students are expected to learn not only mathematical knowledge but also important basis of mathematical problem solving such as mathematical ways of thinking, beliefs, and meta-knowledge of "how to learn."

Here, an example of teaching situation is shown. Mr. Tsubota who is a mathematics teacher in the Elementary School Attached to University of Tsukuba, Tokyo, conducted a class by using the "marble problem" (Tsubota, 1988). The students were in grade 6 (11 to 12 years old). I will illustrate the flow of class according to the three situations described earlier.

Situation A: "Formulating a problem mathematically." The "marble problem" was presented to students not by sentences but by a game situation as follows:

Teacher: We will play a game of throwing marbles on a piece of paper and comparing how much the marbles scattered. Winner is the one whose marbles scattered most. Let's see, each of the three people, A, B and C, threw marbles, and the marbles scattered in this way (teacher shows the students figures). Who do you think is the winner?



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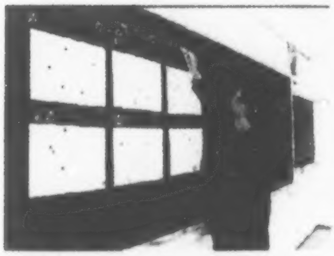
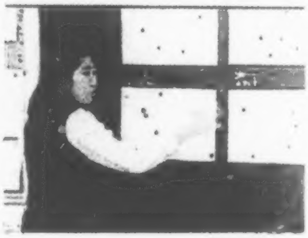
Then, Mr. Tsubota let the students experience the marble-throwing-game by themselves on their desks. After a while, he asked several students to present their results on the blackboard. By looking at presentations by their peers, the students began to realize that there are a variety of scattered ways. Mr. Tsubota then asked the students how to decide which marbles are scattered more and how to convince others that is a reasonable decision. Several students raised their hands and made proposals. Many of them capitalized on their knowledge as sixth-graders and put their focuses on lengths and areas. Discussion was gradually shifted to the differences among proposed ways of making decision.

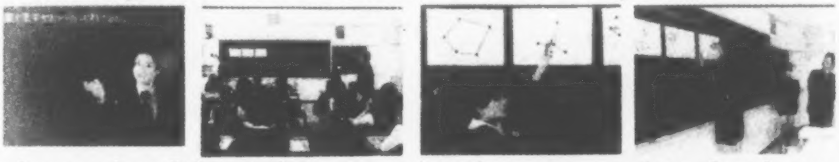




This way of presenting problem is often seen in mathematics teaching in Japan. In the case of Mr. Tsubota, he cultivated students' mathematical words such as "length" and "area" in the earlier part of the discussion. Based on these naturally verbalized words by the students, he then posed the essence of the problem, "Is it possible to use number for making a good decision?"

Situation B: "Investigating various approach to the formulated problem."

In the later part of the discussion, the students came to present different ways of using number. Other students seemed to share strong points of each way. Still, some students proposed counter-examples and indicated that some of the ways cannot be applied to extreme cases. Through intense discussion, the students came to integrate variety of solutions into more sophisticated ones.



Situation C: "Posing advanced problems." After all presentations were made, Mr. Tsubota let the students go back to the original game situation and decide the winner by using someone's proposed way. The students said that they liked simpler ones, and finally decided to use "area" to make comparison. They measured the "areas" of their own scattered marbles and decided the winner. At the same time, it became apparent that in some cases the "area" did not give reasonable measures for the purpose of comparing the degree of scattering. At the end of the class, the teacher and the students recognized that the "area" method needed to be revised further and reflected on today's class. The teacher concluded the class by saying, "Today, we learned how to measure and compare ambiguous objects."

In sum, Mr. Tsubota's class illustrates that it is possible that students (i) pose problems in the problematic situation, (ii) formulate their own approaches by themselves, (iii) accept that there are a variety of solutions, and (iv) closely examine, justify and refute different solutions. It shows that the open-approach method enables the construction of vital mathematical activities in the classroom.

PERSPECTIVES FOR FUTURE MATHEMATICS EDUCATION RESEARCH

Although the theory of open-approach was constructed around 1980, the above discussion shows that it has many contact points with the ideas discussed in the mathematics education community today.

In the open-approach method, teachings are required to be open to student's mind. Such requirement can be found in constructivist approach to teaching, which was raised in mid 1980s. Teachers following the open-approach try to orchestrate their lessons by taking advantage of students' thoughts, even when those thoughts are unexpected for the teachers. This seems closely related to the idea of "learning trajectory" that M. Simon has proposed (Simon, 1995). The openness of approaches to one problem is also an important aspect of the open-approach. The class discusses student's various ideas and thoughts, and develops them mathematically through sophistication by the peer group and appropriate advises by the teacher. Thus, the open-approach class may share the common interest with the class that emphasizes mathematical discussion and communication. Furthermore, the evaluation in the open-approach method, where the emphasis is laid on students' ways of mathematical thinking and their creativity rather than correct answers, reflects the common expectation with the research that facilitate students' attitudes and beliefs in problem-solving oriented classes (Nohda, 1993).

These similarities imply that the recent research findings in these areas can enrich the open-approach method, while the ideas underlain the open-approach method can be used as a global framework for integrating the fruits in the research areas. Considering that the open-approach style is also open to mathematics, the idea of open-approach can present a viewpoint from which we can reexamine how mathematics is located in the research or proposed teaching methods. For example, the open-approach presented some viewpoints in its evaluation, like flexibility, originality, and elegance, which reflect the nature of mathematics or "doing mathematics." This means that it tries to evaluate not only students' positive or active attitudes to mathematics, but also mathematical nature in students' thinking.

The number of people who are interested in Japanese mathematics classes has been increased since the mathematics education reform movements around the world in 1990s. The open-approach method is based on the tradition of Japanese mathematics education community in a sense that it made good characteristics of the tradition explicit and extended them. Therefore, its basic spirit, "be open both to students and to mathematics," can be a perspective for investigating Japanese lessons. In fact, this is consistent with the analysis of TIMSS Videotape Studies. When Stigler and Hiebert characterized the lessons in the US, Germany, and Japan in terms of relationships among students, teachers, and mathematics, they stated that students and mathematics were dominant in Japanese lessons (Stigler & Hiebert, 1999, pp. 25-26). Because of being open to both of students and mathematics, problem solving oriented lessons would result neither in teachers' demonstration of the best solutions nor in mere presentation of students' various opinions.

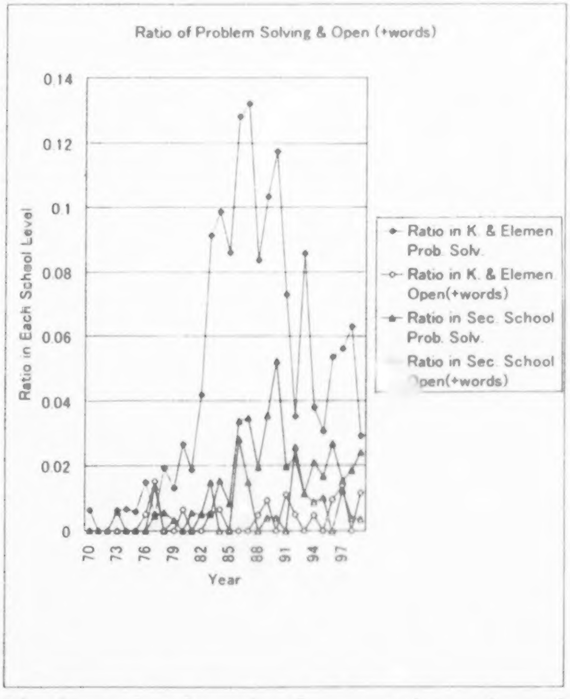
I would like to conclude this paper with two research problems to be studied further. First, we need to develop more good problems, especially open-ended type problems. As stated above, it is most difficult to construct open-ended type problems among the three types of open problems. There is not sufficient stock of such problems even in Japan, so we need to develop them because they are very valuable to mathematics education today. We examined the "marble problem" as the example of open-ended problem. In that problem, students are expected to mathematically make sense of the situation where marbles are scattered. This suggests that some open-ended problems can be related to mathematical modeling. It may be possible to get hints for developing good open-ended problem by referring to the research on mathematical modeling.

Second, we need to study changes in students' mathematical ways of thinking more closely. Many teaching practices following the open-approach were implemented, and the changes in students' understanding and positive attitudes have been explored. But we do not fully understand how each student's creativity and mathematical thinking can develop and what is the cue for such development. Investigation of such issues is also needed for improving mathematics classes through the open-approach method.

NOTE

The table below shows the ratio of presentations at the Annual Meetings of Japan Society of Mathematical Education from 1970 to 1999 that included the words "Problem Solving" or "Open" in their titles by the levels of education. The graph below shows the ratio in each year.

	K. &Ele.	Lower Sec.	Upper Sec.	Tertiary
Total Num. Of Presentation	5246	3586	3213	435
Ratio of "Prob. Solv."	5.4%	2.6%	0.4%	1.1%
Ratio of "Open (+words)"	0.4%	0.9%	0.2%	0



The table shows that the ratio of presentations whose titles include "Open" is high in secondary education levels. It is aligned with the aim of open-approach, i.e., to contribute to a variety of students' differences at these levels. The figure shows that problem solving oriented class was pervaded during 1980s especially in elementary schools. This is consistent with the trend by NCTM at that time.

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Basic Issues for Research in Mathematics Education

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Mathematics education covers a very broad range of topics, from primary school to university. It can be analysed from different points of view, epistemology, psychology... But, whatever topic and point of view may be, research in mathematics education entails theoretical and methodological choices on some core problems about the nature of mathematical knowledge with regard to all other kinds of knowledge. Does it depend on the same thought processes as the other kinds of knowledge or does it require the development of some specific ways of cognitive working ? Can be mathematics learning mainly described as a concept acquisition ? Which kind of representation is relevant in mathematical understanding ? Which field of phenomena can show the conditions of understanding and knowledge acquisition in mathematics ? These problems provide alternative choices. We can assume that thinking works in mathematics like in the other areas or that it works in a very specific way. We can focus either on objects and concepts particular to one mathematical area or on constant features of the mathematical activity. We can also focus either on mental and individual representations or on semiotic systems of representation. We can focus either on class room activity or on individual acquisitions over several years.

These basic issues are not purely theoretical. The choices lead to different ways of specifying relevant variables for mathematics learning, and they do not yield equal possibilities to explain the variety of difficulties that students come against up throughout their studies. From primary school to higher secondary level we can notice a strong contrast beween very spontaneous simple mathematics for every child and a little more advanced mathematics, for example when new concepts are introduced or when algebra is brought into use, when theorem proving is required or when graphs are used in analysis as an obvious tool of visualization... And we can see an increasing gap for learning : more and more students seem to reach a breaking point in their understanding of mathematics. Are we faced with the same kind of phenomena ? More precisely is there something similar in the process of mathematics learning at the first levels and at upper levels ? In fact, because of teaching requirements which are peculiar to each level of study and, also, because of internal evidence of mathematics, for teachers and mathematicians, some choices appear essential and obvious.

However, we must pay more attention to these basic issues, at least if we want to understand deep mechanisms of mathematics learning and difficulties most students encounter throughout their curriculum. Our purpose in this paper is to come back to these basic issues and to explain why our research has progresively led us to choices which are diverging from those considered as essential and obvious. In other words, the main question about mathematics learning is : does mathematics understanding require specific ways of cognitive working in comparison with the other fields of knowledge ? Or, from a phenomenological point of view, do visualization, language and conceptualisation work in mathematics in the same way as in other situations ? If it is not the case, what kind of cognitive working is required in order to understand mathematical objects and processes, in order to become equally able to apply them, and how can any student master it ?

I. Analysis of knowledge acquisition in mathematics education research

(1) What do analysis of mathematics knowledge focus on : concepts understanding or underlying thought processes ?

Mathematics are divided into various areas : arithmetic, geometry, algebra, calculus, statistics... And for each area we have, on the one hand, a set of concepts relative to objects such as numbers, functions, vectors, etc. and, on the other hand, specific algorithms, procedures, methods of problem solving which are close connected to concepts/objects. From preprimary schools to senior secondary schools, students must discover or learn some basic concepts and algorithms with their applications within these various areas... Thus we are faced with a large scope of teaching goals And each one leads to focus on the concepts/objects according to specific problems that their teaching can involve : what kind of situations to introduce them or to justify their introduction, what kind of mistakes can occur, what kinds of progression...? In these conditions, it seems difficult to avoid a certain compartmentalization in research. But above all, what concerns the common deep processes which underlie mathematics understanding are put off investigation. Learning processes are assimilated to the construction of such-and-such a concept.

Whatever the concept/object you choose, mathematics knowledge requires thought processes which are multidisciplinary and typical of what it is to understand in mathematics. That appears through validation, through proving, through using symbols and various visualization forms (cartesian graphs, geometrical figures..). For example, it is usual to observe a gap between the use of words and the use of symbols, between «the use of mathematical expressions and the way they are understood» (Sierpinska 1997 p.10), or between the spontaneaous ways of seeing geometrical figures and the mathematical ones. Learning mathematics is not only to gain a practice of particular concept/objects and to apply algorithms, it is also to take over the thought processes which enable a student to understand concepts and their applications. And these thought processes cannot be assimilated to construction of such-and-such a concept.

In the case of proof learning, that alternative between mathematical concepts/objects side and involved thought processes side appears clearly with proof, one of the most difficult topics in mathematics education. Because the ways to show why a proposition is true are not the same for theorems in mathematics as for statements about phenomena of the external world. How to help students gain insight into these very specific mathematical ways ? And why teaching does not succeed in finding such help with most students ? One can emphasize the need to provide not one but several proof methods or the importance to be confronted with rich epistemological context such as a physics problem ... That requires exploration of a particular set of data and activities for each theorem. But what matters is not only to gain insight why such proposition can be true, but to understand how proving in mathematics works and to gain the thought processes involved in proving. That changes the perspective within the educational problem of proof appears. Why, for example, cannot students really understand mathematical ways or reasoning, whenever natural language is used and whatever the proposition they have to prove ?

(2) From what kind of phenomena can the specific problems raised by mathematics learning be examined ?

In order to study the complexity of mathematics learning, we must take into account the students and not only the epistemological complexity of the taught concepts. But there are many ways to refer to what the students do, to their explanations, to their achievements, etc. We can try to observe live behaviours and productions over the learning time or, on the contrary, evolution of mathematical skills within further various situations over a whole curriculum. We can also focus on individuals or on the activities of the class including the teacher and the teaching organisation, or on the whole population of an age group. Thus, we have several possible areas of observation (both scale of time and field of study). Each area requires a specific methodology *because parameters and variations that can be checked are not at all the same.* And when we change the area of observation the problems of learning appear in an other light.

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Scale of time →

		ongoing learning of mathematical concepts (over a short-lasting time)	**curriculum** (over several years of teaching) **What transfer ?**
Field	Individuals	one or several sessions - inside class activities (*some particular productions*) - outside class activitties (*interview, experimental frame..*)	— feed-back of new acquisitions on the previous learnings — skills that can be mobilized in further situations at a higher level (*transversal or longitudinal methodology*)
of	One particular classroom: the teacher or/and the students ?	one or several sessions (*case study....*)	✕
study	Population of an age group	at the end (*assessment*)	— feed-back of new acquisitions on the previous learnings — skills that can be mobilized in futther situations at a higher level (*assessment*)

Figure 1. The various areas of observation of mathematics learning.

There are many reasons and social demands which lead to emphasize one area rather than the others or to consider one particular kind of phenomena as the most relevant or the most significant. But the problem is not in this heterogeneous range of possible areas. It is about the depth of acquisition and the possibilities of transfer. Where and how can we gain data about this crucial aspect for any learning ?

For that, we must distinguish three kinds of difficulties that students come up against in mathematics learning :

— **temporary difficulties** in order to succeed the local goals of any learning sequence : they depend on degrees of newness for students, on misleading similarities to what is already known, or on the background of the underlying epistemological complexity

— **recurrent difficulties** whenever context is changed (for example, heuristic using of geometrical figures in problem-solving leads to such changes), or whenever new objects are introduced

— **standing or insuperable difficulties** (for most students) : they underlie local ongoing acquisitions and inhibit further acquisitions. They appear whenever students are faced with a proof task or with some verbal problem in arithmetic or in algebra

Hence the following question : what kind of difficulties do we have to examine, if we want to analyse the thought processes which are required for mathematics understanding and therefore the specific conditions of mathematics learning ? Temporary or standing difficulties ? Obviously all kinds of difficulties must be taken into account in teaching. But over ongoing learning and in the field of class activities, they cannot be truly discriminated. And, in fact temporary, difficulties are necessarily uppermost in the didactic purposes of the teachers. And we cannot avoid the question whether results at local scales can be extrapolated at global scales. Anyway when analysis is turned towards temporary difficulties, phenomena relative to epistemological complexity are favoured and, on the contrary, when it is towards insuperable difficulties, phenomena relative to the cognitive functioning of subject become the most significant.

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II. Kinds of representation involved in mathematical thinking

There is no knowedge without representation. But from Descartes until now, through Peirce and Piaget, many changes have taken place in the way to consider the relationship between knowledge and representations, and the nature of representations appears to be more and more complex (Duval article 1998b, Duval *&alii*, 1999). When we talk of "representations" the four following aspects must be taken into account :

— (a) *the system by which representation is produced.* Any representation is produced through a particular system. It can be a physical device such as camera, or an brain organisation as for memory visual images, or even semiotic system such as various languages. And the content of the representation of an object changes according to the productive system of representation which is used. It means content of any representation depends on its productive system and not only of the represented object. The content of a verbal description of a man in order to recognize him is not the same as the content its sketch portrait, or the content of the graph of a function is not the same as the content of its analytic expression. Human thinking require the mobilization of several heterogeneous productive system of representation and their coordination. Do thought processes especially require semiotic systems as the main constituent of the cognitive architecture which enables any individual to understand mathematics ? And, in mathematics education, what is it crucial for learning, (a1) taking into account the global and spontaneous individual state of beliefs about a subject (Peirce, the first Piaget), or (a2) making the students aware of the ways of functioning of the semiotic productive systems which are used in mathematics ?

— (b) *the relation between representation and the represented object.* There are two kinds of productive systems of representation : on the one hand physical devices and neuronic organisations, on the other hand semiotic systems. In the first kind (b1) (physical and mental images), the relation is based on action of an object on the system (causality), and in the second one (b2) (words, symbols, drawings) the relation is only denotation. In mathematics education, when we talk of "mental images" what kind of relation are we referring to ?

— (c) *the possibility of an access to the represented object apart from semiotic representation.* We have representations (c1) which are an evocation of what has already been perceived (Piaget 1926, 1946) or what could be perceived and representations, or (c2) about objects (mathematical objects) which cannot be perceived.

— (d) *the reason why representation using is necessary* : either (d1) mainly communication or (d2) processing (computation or discursive expansion (Frege 1891, 1892), anamorphosis, etc).

According to the way these aspects are taken into account, what is referred as representations change. I will confine here to the relevant issues for mathematics learning.

(3) Which brings about the most misunderstanding : subjective representations of students or manifold semiotic representations used in mathematics ?

Many studies have examined students mistakes over the learning of concepts for each level and some failures remain whatever teaching method is adopted. In order to explain these structural misunderstandings, subjective representations (a1, c1) are emphasized as being the root of obstacles encountered over learning. Thus, in the triadic conceptualisation of Peirce (2.228) {Object, "representamen" (sign), "interpretant"}, interpretant is emphasized in such a way that representations are mainly mental phenomena and individual beliefs.

Progress in mathematics has involved development of several semiotic systems from the primitive duality of cognitive modes, image and language, which are linked with the more informational sensory receptors : seeing and hearing. For example, symbolic notations stemmed from written language and have led to algebraic writing. For visualization, the construction of plane figures with tools, then that of figures in perspective, then that of graphs in order to "translate" curves into equations. Each new semiotic system provides new means of representation and processing for mathematical thinking. So that for any mathematical object we can have different representations produced by different semiotic systems (a2, c2). Thus we must change the triadic conceptualisation of Peirce in the following way : {Object, one of the various semiotic systems, composition of signs). But that necessary variety of semiotic systems raises important problems of coordination.

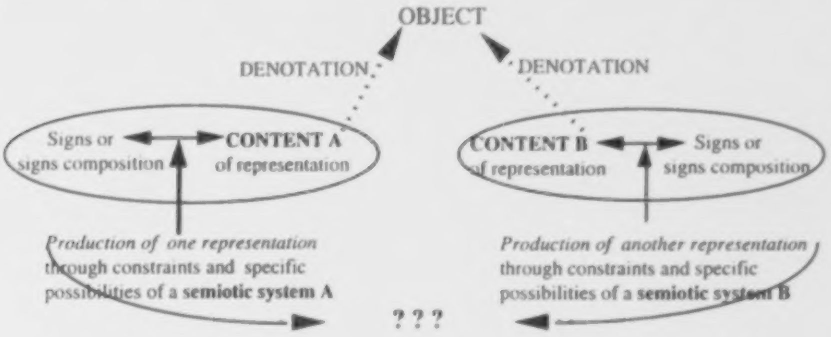
OBJECT
DENOTATION
DENOTATION
Signs or signs composition
CONTENT A of representation
CONTENT B of representation
Signs or signs composition
Production of one representation through constraints and specific possibilities of a semiotic system A
Production of another representation through constraints and specific possibilities of a semiotic system B
? ? ?

Figure 2. Representation and understanding for mathematical knowledge

In that perspective, deeper causes of misunderstanding appear. Whenever a semiotic system is changed, the content of representation changes, while the denoted object remains the same. But as mathematical objects cannot be identified with any of their representations, many students cannot discriminate the content of representation and the represented object : objects change when representation is changed!

Here the issue is to know what kinds of representation is crucial for mathematics learning. Emphasizing individual beliefs, as for physical phenomena (Piaget), leads to assume a purely mental cognitive model in order to analyse acquisition of mathematics knowledge. And semiotic representations are considered as external to thought processes. Is such an assumption obvious and, above all, relevant ?

(4) Is the distinction between mental and material representations relevant for the use of semiotic representations in mathematics knoledge ?

This distinction is based on three considerations. First, the dualistic opposition, for any sign, between signifier and meaning, between what must can be perceived and what is evoked in the mind (c1). Then understanding is about objects and goes beyond the content of any semiotic representation. Lastly, semiotic representations are needed for communication (d1). Hence the opposition between purely mental representations which would be enable anybody to understand and semiotic representations which would be mainly for communication and social interactions. And it is often argued that semiotic representations used by somebody else are sometimes difficult to understand.

However, in mathematics, semiotic representations does not fulfill first a communication function but a processing function (d2). It is only through semiotic representation that mathematical numbers can be reached and used. Progress in the human numbers knowledge has been closely connected with progress in numeral systems. In fact, the opposition between mental and semiotic systems is deceptive because it is the outcome of the confusion between two heterogeneous aspects in representation production : the phenomenological mode and the used system. Moreover, in external phenomenological mode, we must distinguish oral and visual (writing, drawing) modes. Semiotic representations are neither mental as images of memory (b1) nor material as pieces which can be physically handled.

Phenomenological MODE of production

		Mental	**Material**	
			oral	**visual (writing, drawing)**
Kind of SYSTEM of production	**SEMIOTIC (intentional production)**	*objectivation and processing functions*	***communication** functions*	***processing**, communication objectivation functions*
	NATURAL (automatic production)	*objectivation function (mental imagery)*		

Figure 3. Production of representation and relationships between thinking, semiotic system and the main cognitive functions.

Semiotic systems of representation play an essential part in all the main cognitive functions : not only communication but also processing, that is the transformation from one representation into another one inside the same system and without resort to further external data. And semiotic reprentations too are necessary to enable anyone to become aware of something new. Objectivation is an expression or representation for himself, which can be either mentally or materially produced. But its constrainsts are quite different from these one in social interactions.

The way we take into account semiotic representations involves an implicit model of cognitive working of human thought and entails choices concerning mathematics learning.

Thus one can really wonder why pure mental models, that is off-semiotic representation models, are always postulated to explain mathematics understanding. What seems simple or purely mental in the inner evidence of understanding, especially when you have become an expert, is the outcome of a very long process of internalization of semiotic representations.

One can also ask whether the variations of apprehension between the oral mode of production and the visual mode of production do not lead to introduce a distinction between two kinds of mathematics : spontaneous mathematics which can be discovered, or done, by everyone, child or adult, at school or ouside school, and hardly more advanced mathematics which require, on the contrary, skills in extensive processing of semiotic representations. So that the jump in learning would be between mainly oral practice of mathematics and necessarily writing practice of mathematics. The passage from additive to multiplicative operations, or this one from natural numbers to decimal numbers seem to require such a change of practice. But also proving in the discovery of which writing can be a necessary stage for purpose of objectivation and not only of communication (Duval 1999).

III. What kind of model is relevant to explain the mathematics learning process ?

Somehow, any model must refer to the organisation of a field of phenomena and describe its way of working. With regard for the mathematics learning, we can distinguish two great kinds of models : the developmental models and the cognitive models.

The developmental models focus on the increase in knowledge. Initially they referred first to two fields of phenomena. On the hand, the historical ways whose mathematical concepts/objects were discovered and on the other hand the ways in which young children become aware of natural numbers, geometrical shapes, schematic representation of environnemtal space... And a relative parallelism was postulated between these two fields of phenomena in order to explain acquisition of

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mathematics knowledge. Thus there a link was established between epistemology and developmental psychology. On this basis, constructivist model of development appeared as the sketch description of every acquisition of mathematics knowledge. And therefore learning processes over the curriculum would have to follow constructivist "laws" of knowledge acquisition

Developmental models have led to enhancing a third field of phenomena, the interactions between students inside classroom, especially while they are solving problems. These interactions present three advantages : they correspond to a main factor in the constructivist model of acquisition, they can be managed by the teacher, they enable researchers to observe live learning processes (see above I. 2).

In a developmental model, explanation of learning process is refered to common schemes which would underlie any increase in knowledge at historical scale of discoveries, at genetic scale of child's intelligence growth (outside of any teaching from the Piaget's view point) and also at the local scale of group work. The cognitive complexity which underlies mathematics understanding is not taken into account except for subjective representations when they seem to hinder learning (see above II.3).

The cognitive models focus on the cognitive complexity of the working of human thought. At first sight, they seem far from mathematics learning. And classical models developed in psychology laboratories cannot be used as they are (Fischbein, 1999). By the simple reason that the learning of mathematics raises specific and fundamental questions about reasoning modes, about the treatment of figures, about the understanding of mathematical concepts —and infinity is a very important instance— which are not envisaged by psychologists. Nevertheless, there is a core question which cannot really be raised in the framework of the developmental model :what are the ***internal cognitive conditions required*** in order that ***any student can understand*** *mathematics at any level of primary or secondary school ?* Note that we are talking now of "understanding" and not only of "learning". These internal cognitive conditions refer to what was called the archictecture cognitive , that is an organisation of several systems (Kant, p.619) : in such an archictecture several semiotic systems must be included or more precisely incorporated into natural systems.

We have already evoked two important facts. Whatever the phenomenological mode of production of representations, working of human thought involves using one or several semiotic systems : the first of all is the native language. But acquisition of mathematics requires other semiotic systems such as the decimal numeral system, algebraic writing or formal languages..; which are suited to mathematical operations. Unlike oral native language, the semiotic system used in mathematics as well as written language, are not natural. In the context of the core question, research on learning processes, must take into account how such semiotic systems can be internalized by students and under what conditions they can become operative for each student on.

The alternative bewteen developmental models and cognitive models concerns directly the way the problem of mathematics learning is raised and analysed : either an increase of knowledge according to common and general processes or a minded-opening to quite specific thought processes.

IV. The paradoxical character of mathematical knowledge

Concerning the cognitive mode of access to objects, there is an important gap bewteen mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary, as for any object or phenomenon of the external world. We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. *This conflicting requirement* makes the specific core of mathematical knowledge. And it begins early with numbers which do not have to be identified with digits and the used numeral systems (binary, decimal).

Obviously, it is not a significant characteristic for mathematicians and epistemology does not take it really into account. From an intrinsic mathematical point of view the semiotic side, which is the only directly accessible, seems to be transparent or subsequent to non-semiotic actions. But from a comparative epistemological point of view, the conflicting requirement cannot be erased. On the contrary it appears as the crucial problem of mathematics learning. In the other fields of knowledge, semiotic representations are images or descriptions about some phenomena to which we can gain a perceptive or instrumental access, ouside any semiotic representations. In mathematics it is not the case. In these conditions, how can a student *learn to distinguish* a mathematical object from any particular semiotic representation ? And therefore, how can a student *learn to recognize* a mathematical object through its possible different representations ? At every level, among many students, inability to convert a representation from one semiotic system into a representation of the same objet from another system can be observed as if both representations refer to two different objets. This inability underlies the difficulties of transfer of knowledge and also the difficulties to translate verbal statements of any problem into relevant numerical or symbolic data for mathematical solving.

This conflicting requirement, which is typical of mathematical knowledge, can be approached otherwise. It is very often assumed that mathematics resort to the most common thought processes : reasoning and visualization. And this assumption is particularly strong in the teaching of plane and solid geometry. But there teaching comes up against difficulties which indicate *an imperceptible but deep gap between common thought processes and mathematical processes.* Considering always persistent understanding blocks about theorems proving and heuristic using of geometrical figures in problem solving is enough to ask questions about the specific cognitive working that mathematical knowledge requires.

The recurrent confusions between hypotheses and conclusion, between a statement and its reciprocal, and other dysfunctions are only the expression of the natural discursive practice in the ordinary way of reasoning. In fact, under similar pratices of speech, there is a discrepancy between the kind of organisation of propositions within a valid reasoning and the one in any common argumentation or explanation (Duval to be published). In order to make students become aware of this discrepancy a cognitive detour is required (Duval 1991). Understanding what is being proved in mathematics is not at first a matter of learning methods, facing different proofs for the same theorem or even convincing other students...

Nothing seems more obvious than a geometrical figure. It seems providing directly to see, even if every figure is always a particular configuration. In fact when the goals of teaching go beyond recognizing or constructing elementary cultural shapes, the gap between figures perception and mathematical way of seeing figures is widening. Mathematical visualization, in the case of geometrical figures, leads away from any iconic representation of physical shapes. Unlike iconic representations, figures are not sufficient to know what are the denoted objects (Duval 1998a). Besides, for the same object, we can have quite different possible figures : thus, for example, there are two typical figures for a parallelogram and only one is iconically a paralelogram shape. But when it is a matter of solving a geometrical problem, the complexity of using geometrical visualization increases fast for most students, even at upper levels. And there we are faced with a field of phenomena which cannot be explained only by the epistemological complexity of such-and-such a concept !

We can focus on the paradoxical character of mathematical knowledge or put it on the fringe. That means to emphasize what is specific to cognitive working in mathematics understanding or to confine cognitive structures that would be common to any kind of knowledge. That means also either to take a comparative viewpoint with other fields of knowledge or to take one only within mathematics. In order to study mathematics learning, we must take into account mainly the insuperable difficulties. And these difficulties, which are the most inhibiting for students, must be analysed in relation to the conflicting requirement and to the gap between common thought processes and mathematical processes. Which raises the following question : what is the cognitive working that underlies understanding in mathematics ? And that leads to highlighting the importance of representations not in the ordinary sense (a1, c1, d1) but in the alternative one (a2, b2, c2, d2).

V. The cognitive working that underlies understanding in mathematics

We cannot talk about representation without relating it to its system of production. But to take into account semiotic systems means focusing on **the transformations** of representations. Thus we must distinguish two kinds of transformations : "processing" and conversion.

Some semiotic systems provide specific possibilities of intrinsic transformations of representation. Any transformation produced in one system can be changed in another representation of the same system. Thus, paraphrase, réformulation, computation, anamorphosis, reconfiguration, etc. are transformations of semiotic representations which can be achieved only in one specific register. We referred to this kind of transformation as "processing" and we referred to semiotic systems which provide such possibilities as registers of representation.

For any representation of an object which is produced within a system, we can also produce another representation of this object into another system. We referred to this kind of transformation as conversion. Thus constructing a graph from a given equation or writing an equation from a graph, translating a verbal statement into a litteral expression or into a equation... Geometry is a field where conversion is very much in demand, as well implicitly as explicitly. But numbers required also changes of representation which are more similar to conversion than to processing, even with the simple change from decimal expressions to fractionary expressions, apart from a few frequent associations such as 0,5 into 1/2.

Researchers do not pay very close attention to **the gap between these two kinds of cognitive operations.** In mathematics processes and in analysis of mathematical tasks, they are not really separated, whenever they are implicitly or explicitly needed. They are looked upon as a whole. For example mathematical activity, in problem solving situations, requires the ability to change register, either because another presentation of data fits better an already known model, or because two registers must be brought into play, like figures and native language. From a cognitive view point the real problem is to know whether these two kinds of transformations can be considered as depending on the same deep thought processes. All observations show that is not the case.

a. The irreductible cognitive complexity of conversion

Conversion is the transformation of representation of an object by changing register. Two main facts can be observed at any level.

In some cases conversion is obvious and immediate as if the representation of the starting register is transparent to the representation of the target register. In other words, conversion can be seen like an easy translation unit to unit. Conversion is congruent :

set of points whose ordinate is greater than abscissa $\longrightarrow y > x$

In other cases it is just the opposite. Conversion is non-congruent :

set of points whose ordinate **and** abscissa are **with the same sign** $\longrightarrow x \ (\times) \ y > 0$

Non-congruence is the crucial phenomenon for any task of conversion. Difficulties and mental blocks stem often from the inability to achieve a conversion, or to recognize it when it has been made. But what is the most surprising with this crucial phenomenon for mathematics understanding is its unidirectional character. A conversion can be congruent in one way and non-congruent in the opposite way. Congruence or non-congruence are closely connected to the direction of conversion. That leads to striking, typical and particularly persistent contrasts of performances, such as in the following figures.

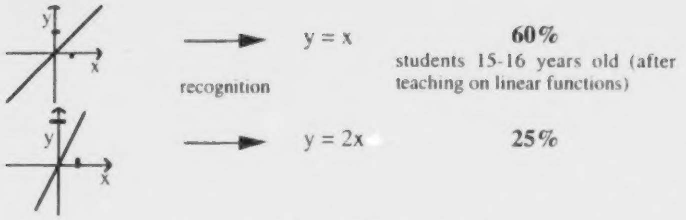

y
x
recognition
y = x
60%
students 15-16 years old (after teaching on linear functions)
y
x
y = 2x
25%

Figure 4. (Duval 1988, 1995b)

Obviously, in the opposite direction, conversion is very easy and there is no more difference between equations (Duval 1996b) . And at higher level we find the same analogous results.

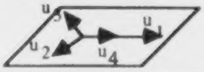
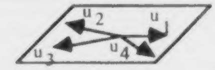
	Starting register	**Target Register**	144 students
(2D Rep.) **T → G**	1 0 k p 0 1 m 0		**. 83**
G → T		1 0 a c 0 1 b d	**. 34**

FIGURE 5. Elementary task of conversion (Pavlopoulou 1993, p. 84)

By bringing into play systematic variations, contrast of successes and failures for the same mathematical objects appear in similar situations! Very accurate analyses of the congruent or non-congruent character of the conversion of a representation into another one can be systematically done. And they explain in a very accurate way many errors, failures misunderstandings or mental blocks (Duval 1995b, pp. 45-59; 1996a, pp. 366-367).

For every couple of registers, typical facts such as these can be sytematicaly observed. What do they mean ? We can see that two representations of an object do not have the same content from a register into another. And when conversion from one into the other is non-concgruent, the two contents are understood as two quite different objects. Students don't recognize it anymore. And there are good reasons for that. The apparent lack of correspondence between two contents of representations of the same object stems from the fact that content of representation does not depend first on the represented object but on the activated system of production. That means not only each register provides some specific possibilities of processing, but also does not explicit the same properties of objets as the other registers.

Now we are coming up against the consequences of the paradoxical character of mathematical knowledge. Since there is not direct access to objects apart from their representations, how can a student learn to recognize a mathematical object through its various possible representations when their contents are so different ? Explaining that as a lack of conceptual understanding is not a right explanation because we have reversals of successes and failures when changing the direction of conversion. In fact the explanation must be searched at a deeper level. Failures or even mental blocks when conversion is non-congruent reveals **a lack of co-ordination between the registers** that have to bring into play together. And if we come back to the schema (figure) we see that conceptual understanding is possible when such a coordination is achieved. Because of this, the condition for mathematical objects are not confused with content of representation. We can complete the above schema (figure 2) in the following way :

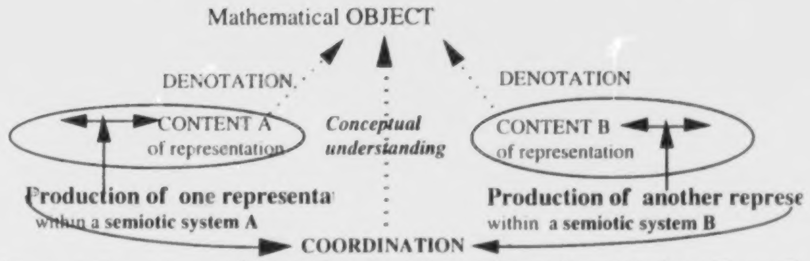


Figure 6. Cognitive conditions of mathematics understanding (see above Figure 2)

But for most students, understanding in mathematics is "confined to some processes within strongly "compartmentalised" registers. Learning mathematics consists in developing progressive coordinations between various semiotic systems of representation.

b. The cognitive ambiguity of some kinds of processing

We must remind that processing is a transformation of representations within one particular register.That means : ways of working of processing do not depend only on the mathematical properties of objects but also on the possibilities of used register. For example we have not the same process of computation with decimal and fractionary notations :

0, 25 + 0, 25 = 0,5	1/4 +1/4 =1/2
0, 5 : 0, 25 = 2	1/2 : 1/4 = 4/2

And we must also distinguish **multifunctional registers** from **monofunctional registers.** Multifunctional registers are those used in all fields of culture. They are used as well for communication goals as for processing goals. And, above all, they provide a large range of various processings. Thus natural language is necessarily used in mathematics but not with the same way of working as in everyday life (Duval 1995b, cap.II). Within these multifunctional registers, processings cannot be performed or changed in a algorithmic way. On the contrary, monofunctional registers have been developed for one specific kind of processing, in order to have more powerful and less expensive perfomances than those within multifunctional registers. Here processing becomes technical and using signs or expressions depends first on their form. Technical processing are formal processing. That's why processing can be expanded as algorithms.

	DISCURSIVE REPRESENTATION	NON-DISCURSIVE REPRESENTATION
MULTIFUNCTIONAL REGISTERS : **non-algorithmisable processings**	natural language *verbal (conceptual) associations,* *reasoning (argumentation from observatins or beliefs, valid deduction from definitions or theorems...)*	geometrical figures as shape configurations, plane or in perspective *operative apprehension and not only perceptual apprehension* *construction with tools*
MONOFONCTIONAL REGISTERS: **processings are mainly algorithms**	numeral systems symbolic or algebraic notations, formal languages *computation*	cartesian graphs *change of coordinates system, interpolation, extrapolation*

Figure 7. Classification of the four kinds of register used in mathematics processes

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Mathematical processes involve at least two of these four kinds of processing as we can see it in any problem solving or in some fields like geometry. Mathematics understanding require the coordination between at least two registers of which one is multifunctional and the other monofunctional. Classic *problématiques* of relations between mathematics and language can be put in an accurate and relevant way only within such a framework of cognitive working. Now if we consider the most advanced level of teaching, the predominance of discursive monofunctional registers seems to increase. Besides it is with this kind of register that both performances and loss of meaning is very often observed. Why? It is wrongly believed that application to daily life or to extramathematical situations can be a source of meaning and therefore of understanding. No! The main problem is first with the multifunctional registers. They are implicitly and explicitly needed for mathematics understanding, but the way they are working in mathematics thought processes is quite different from the one they are working in the other fields of knowledge and, a fortiori, in everyday life. Therefore resorting to natural language as within ordinary speech and referring to geometrical figures as if they were as obvious as other visual images does not help but increases the confusion in understanding and learning. Here a wide field of reseach is opening. If we want to understand the complex mechanisms of mathematics learning we must analyse the specific ways of working of the multifunctional registers, especially for what matters reasoning in proof and visualization in solving geometry problems. We can have already very specific and decisive cognitive variables (Duval 1995a, 1995b, 1996a)

c The cognitive architecture that underlies understanding in mathematics

That quick overview of the complexity of all kinds of semiotic transformations involved in mathematical processes sends us back to the above question : what are the internal cognitive conditions required in order to any subject can understand mathematics ? Now, psychological models of information processing have highlighted that conscious understanding depends on the automatic (unconscious) working of the organisation of various and heterogeneous systems. This organisation makes up the cognitive architecture of the epistemic subject. But mathematics understanding requires a more complex organisation, including semiotic systems, because it depends on the mobilisation of several registers. In these conditions learning mathematics means : integrate into its own cognitive architecture all needed registers as new systems of representation.

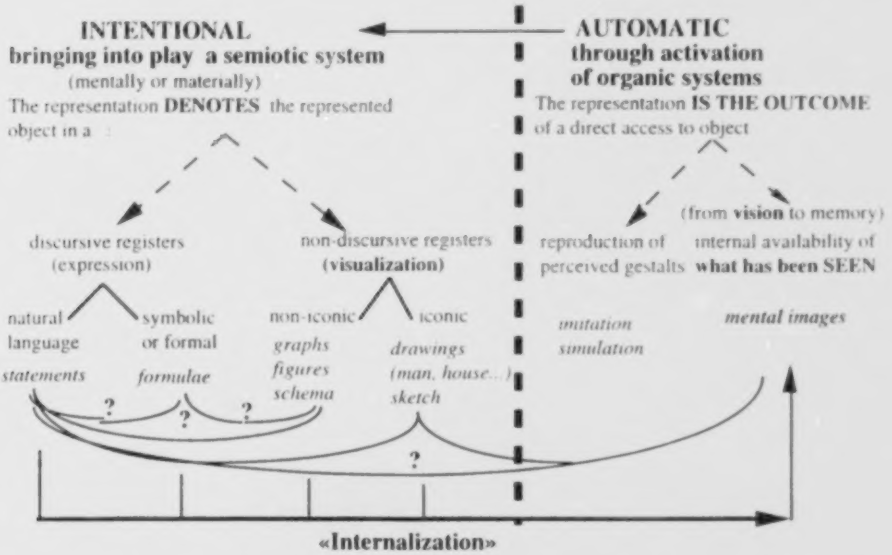
INTENTIONAL
bringing into play a semiotic system
(mentally or materially)
The representation DENOTES the represented object in a :
AUTOMATIC
through activation of organic systems
The representation IS THE OUTCOME of a direct access to object
discursive registers (expression)
non-discursive registers (visualization)
(from vision to memory)
reproduction of perceived gestalts
internal availability of what has been SEEN
natural language
symbolic or formal
non-iconic
iconic
statements
formulae
graphs figures schema
drawings (man, house...) sketch
imitation simulation
mental images
?
?
?
?
«Internalization»

Figure 8. Various coordinations between productive systems required for mathematics understanding

That diagram gives a very simplified presentation of cognitive architecture. For example, we should have to distinguish, for the native language, between common working in social interactions and theoretical working in knowledge areas which are ruled by proof requirement. But it shows the various cognitive coordinations that mathematics understanding requires. Learning mathematics involves both incorporation of monofunctional registers and differentiation of the possible different ways of working within multifunctional registers. But it is not enough. Learning mathematics involves their coordination, or their decompartmentalization. Otherwise, conversion between non-congruent representations will be inhibited. And that is not a side-problem, because registers are non-isomorphous and because showing together two different representations of the same object, in order to create associations, does not work. Learning mathematics is learning to discriminate and to coordinate semiotic systems of representation in order to become able of any transforming of representation.

That can summed up in one sentence. Mathematics learning does not consist first in a construction of concepts by students but in the construction of the cognitive architecture of the epistemic subject. What is at stake in mathematics education through particular content acquisition is the construction of this architecture, because it creates future abilities of students for further learning and for more comprehensive understanding. But this deep aspect is misunderstood because student's individual consciousness, with its beliefs, evidences and interests, is often mistaken for the working of thought processes.

Conclusion

Research in mathematics education is extremely complex, because it must be lead through strained relationships between two heterogeneous kinds of organisation and requirements for knowledge, the mathematical one and the cognitive one. And when we are going from preelementary levels to secondary levels, the predominance from one to the other seems progressively to be reversed. In these conditions, what does research about mathematics learning processes mean ? Are we not confronted with quite different topics, each demanding a particular model ? And would the only common process which could be extracted not be useful mainly in order to organise activity sequences the in classroom ?

In an overview of some basic issues, we have emphasized what in mathematics knowledge is deeply different from other areas of knowledge, rather than what is common. This choice can amaze. Since Piaget's developmental models and also because mathematics are considered as an intellectual subject and are needed in all fields of science and technology, we are inclined to assume common roots between mathematical processes and common thought processes. That is both right and false. It is right because these common thought processes depend on the working of the semiotic system of representation. It is false because the taught mathematics require a more systematic and more differentiated use of semiotic systems than the one needed for anyone who remains at an only oral culture stage, or than the one needed in other fields of culture which do not all resort to mathematics. And thus by highlighting the intrinsic role of productive semiotic systems in mathematics understanding, we emphasize at the same time the gap between natural representations (visual memories, mental images...) and semiotic representations. As we have already said (Duval (Fischbein)) the psychological approach to these fundamental questions requires specific models, which by their turn could contribute to develop the field of cognitive psychology.

In that perspective, conversion of representations and all manifold aspects of non-congruence appear as the typical and basic characteristic of mathematical thought processes. Through conversion we are coming to the core of mathematics learning problems. Furthermore conversion provides a powerful tool of analysis of what is relevant in the content of any representation, because representation is not only considered in itself, but in relation to another register. Thus we can bring out cognitive variables, and not only structural semiotic variations, which determine each register working. It is mainly needed with the multifunctional registers. And by taking into account the two kinds of representation transformations, processing and conversion, a cognitive analysis of problems and exploration of their variations become possible. Unlike mathematical analyses which are downstream analyses, back from various ways of their mathematical solving, cognitive analysis are upstream analyses, from variations of initial conversions forward which can be required in order to start up processing.

All that can seem very far from teaching and especially from questions a teacher can ask in his/her classroom. This deliberate distance reflects the difference between subjective repesentations of individuals and the deep cognitive architecture to construct in order to understand mathematics concepts. The theoretical choices we have made and the model of thought processes that we are developing can lead to many experiments, to other theoretical frameworks, and even in classrooms! But a quite different learning environment than the one of the classroom is becoming more and more important. It requires however the conception of dynamic software which provides very open interactions with learners in order not to be only an assistance for some algorithms learning. The model based on registers of thought cognitive working can be helpful for such a conception and mainly for very complex learning : proving (Luengo 1997) and decimal numbers (Adjiage 1999). In mathematics education issues relative to learning cannot be subordinated to those relative to teaching, because they depend first on the complex cognitive working involved in mathematics understanding. A wide field of reseach is opening ahead of us.

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PLENARY PANEL

Theme:

Teaching and learning in school mathematics:
What has research told us about
mathematics teaching and learning ?

Chair:	Peter Sullivan
Panelists:	Paolo Boero
	Margaret Brown
	Fou-Lai Lin

Physicians leave education researchers for dead

Diane Ravitch is a research professor at New York University in New York City. She was the US assistant secretary for educational research from 1991 to 1993. This was published in the Sydney Morning Herald on 22/2/99 (retyped).

It was an ordinary trip to California? or so I thought. I had taken long weekends to the West Coast many times before, but this time was very different. The difference revealed itself on the morning after my return to New York City: I could barely draw a breath. Some corner of my brain thought "exhaustion", or "prelude to a bad cold", and I decided to ignore whatever was happening.

Twenty-four hours after my return home, my left leg began to ache. Unable to sleep, I got up the next morning convinced that I had a really bad cramp. Ignore it, I decided, because I had to get through the work on my desk and get ready for a trip to Dayton, Ohio, and Chicago later in the week.

After a day at my computer, I could barely stand on the left leg, but my dog forced me to leave the house: she had to go out for a walk. I dragged myself outside and fortunately ran into my neighbour, a radiologist, who happened to be on his way to a community meeting. I ask him whether to put hot or cold compresses on my leg; by chance, he noticed that I was short of breath. He told me to call my doctor immediately.

He recognised the classic symptoms of something I knew nothing about: pulmonary embolisms. The rest of the story is quickly summarised: I went to the emergency department of the local hospital, where my neighbour's diagnosis was quickly confirmed.

I had blood clots in my left leg and in both lungs. If I had not received prompt treatment, the doctors said, I might have died.

When I was in the intensive care unit, the hospital's specialists gathered around my bed, explaining the diagnosis and treatment of pulmonary embolisms to other doctors, residents, and interns. The head of pulmonary medicine described the tests that had been used to ascertain my illness, and the drugs and protocols that were employed to stabilise the clots.

As I lay there, listening to them discuss my condition, I had a sudden insight. I was deeply grateful that my treatment was based on medical research, and not education research. At first, I thought, that's a silly idea, you can't treat pulmonary embolisms with education research anyway. But as the conversation continued literally over my prone body, employing a vocabulary that I did not understand, I began to fantasise about being the subject of education researchers.

The physicians who hovered over me dissolved, replaced in my mind's eye by an equal number of education experts. The first thing that I noticed was the disappearance of the certainty that the physicians had shared. Instead, my new specialists began to argue over whether anything was actually wrong with me. A few thought that I had a problem, but others scoffed and said that such an analysis was tantamount to "blaming the victim".

Some challenged the concept of "illness", claiming that it was a social construction, utterly lacking in objective reality. Others rejected the evidence of the tests used to diagnose my ailment; a few said that the tests were meaningless for females, and others insisted that the tests were meaningless for anyone under any circumstances.

One of the noisier researchers maintained that any effort to focus attention on any individual situation merely diverted attention from gross social injustices: a just social order could not come into existence, he claimed, until anecdotal cases like mine were not eligible for attention and resources. Among the raucous crowd of education experts there was no agreement, no common set of standards for diagnosing my problem. They could not agree on what was wrong with me, perhaps because they did not agree on standards for good health.

Some maintained that it was wrong to stigmatise people who were short of breath and had a really sore leg; perhaps it was a challenge for me to breathe and to walk, but who was to say that the behaviour I exhibited was inappropriate or inferior compared to what most people did? Some people who were short of breath and had sore legs were actually happier, I learned, than people who did not exhibit these traits.

A few researchers continued to insist that something was wrong with me; one even pulled out the results of my CAT-scan and sonogram. But the rest ridiculed the tests, pointing out that they represented only a snapshot of my actual condition and were, therefore, completely unreliable, as compared to longitudinal data (which of course was unavailable).

I was almost completely convinced at that point that the discord among the experts guaranteed that I would get no treatment at all, but then something remarkable happened. The administrator of the hospital walked in and said that she had received a large grant from the Government to pay for treatment of people who had my symptoms.

Suddenly, many of those who had been arguing that nothing was wrong with me decided that they wanted to be part of the effort to cure me.

But, to no-one's surprise, the assembled authorities could not agree on what to do to make me better. Each had his own favourite cure, and each pulled out a tall stack of research studies to support his proposals. One group urged a regimen of bed rest, but another said I needed vigorous exercise.

One prescribed a special diet, but another said I should eat whatever I wanted. One recommended drug X, but another recommended drug Not-X.

Another said that it was up to me to decide how to cure myself, based on my own priorities about what was important to me. Just when I thought I had heard everything, a group of newly minded doctors of education told me that my body would heal itself by its own natural mechanisms, and that I did not need any treatment at all.

My head was spinning with all this contradictory advice. The room turned a few times, and I thought for a minute that I was in that house that got carried away by a twister in The Wizard of Oz. Then, to my amazement and delight, I realized that I was back, safe and sound (but very sick) in my bed in the intensive-care unit at Long Island College Hospital.

I looked appreciatively at the medical doctors around by bed, grateful to be surrounded by men and women who have a common vocabulary, a common body of knowledge, a shared set of criteria, and clear standards for recognising and treating illnesses.

They have access to reliable tests that tell them what the problem is, and they agree on treatments that have been validated over a long period of time.

The thought occurred to me that educators have something to learn from physicians.

Medicine, too, has its quacks and charlatans. But unlike educators, physicians have canons of scientific validity to protect innocent patients from unproven remedies and specious theories. To be sure, not every important question can be resolved by scientific research, but medicine seems to have done a good job of identifying and implementing those that can.

I am grateful indeed that my diagnosis and treatment were grounded in solid medical research. Otherwise, I would not be here to tell my tale.

In our society, we rightly insist upon valid medical research: after all, lives are at risk. Now that I am on the mend, I wonder: Why don't we insist with equal vehemence on well-tested, validated education research? Lives are at risk here, too.

CAN RESEARCH IN MATHEMATICS EDUCATION BE USEFUL FOR THE TEACHING AND LEARNING OF MATHEMATICS IN SCHOOL? AND HOW?

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1. The metaphor of medical sciences

When examining the usefulness of education research results for overcoming serious difficulties in learning, Diane Ravitch adopts the metaphor of treating a serious illness (like pulmonary embolisms) with the tools of medical science. In my view this metaphor is misleading, given that the most serious difficulties in learning, particularly in learning mathematics:

- are of a systemic, cultural nature (while pulmonary embolisms concerns one individual's body). In mathematics education we cannot isolate "learning" from "teaching", nor "learning mathematics" from socially "situated" intellectual and cultural development, including linguistic competencies, metacognitive aspects, rational attitudes, etc.;

- cannot be measured in an objective way (like blood pressure). Objective measurements are suitable for dealing with technical skills, like performing additions; on the contrary complex arithmetic problem solving or conjecturing in the geometry field escape present criteria of objective, quantitative assessment (cfr. Boero & Szendrei, 1998, Section 3);

- must be approached at two levels - the level of autonomous performances, and the level of performances which are potentially attainable with the help of more competent people (according to Vygotsky's idea of "*zone of proximal development*", ZPD). This distinction is significant if we wish to consider "diagnosis" and "remedial interventions" (which must be carried out within the ZPD). No equivalent to ZPD exists for pulmonary embolisms.

All these characteristics which fall outside the metaphor of a serious physical illness (like pulmonary embolisms) raise another problem - that of the field of sciences which mathematics education (ME) belongs to. The problem of the usefulness (and reliability) of results in ME bears strong similarities to the same problem for economics (or even better for some areas of the medical sciences like psychiatry or psychoanalysis, which are very far from the areas involved in the treatment of pulmonary embolisms).

Examining the comparison with other sciences in greater depth, we may say that ME shares with some human sciences the characteristics of a "speculative" discipline (whose criterion of validity is the value of the descriptions and interpretations of what happened or happens, such as in the field of history). On the other hand, it shares with other disciplines the characteristic of a potentially "applicable" discipline (whose criterion of validity consists in the validity of predictions for successful decision-making, such as in the field of economics).

To say that ME belongs to the human sciences does not mean that we must accept the present status of unreliability of many ME results; Diane Ravitch' paper challenges us to find out possible reasons for unreliability of ME experimental research results, and how to improve their reliability.

2. Some reasons behind the unreliability of ME experimental research results

I will refer to the analysis made in Boero & Szendrei (1998) about the reasons behind the unreliability of some mathematics education results.

First of all, the very desire to equate ME experimental research with experimental research in the field of natural sciences can produce negative effects on the reliability of results. If we state that ME experimental research (like experimental research in physics, biology, chemistry, etc.) must produce data that can be reproduced under similar conditions, and whose interpretations may be partly or totally falsified by further research, then researchers must isolate variables, keep experimental conditions under control, etc. But mathematics education is a complex process and research results are interesting (both in themselves and in terms of their consequences for the school system) when complexity is taken into account. This 'complexity' mainly concerns the fact that almost every meaningful teaching-learning process appears as a long-term process involving a large number of interrelated variables; moreover, at present, isolating and measuring some of the significant variables would appear to be fairly difficult.

Another negative effect stemming from the desire to adopting a paradigm similar to those of the natural sciences concerns the importance attributed to *quantitative, objective assessment data* and their impact on the school system. Generally speaking, results of the 'quantitative information' type are very popular among school teachers and administrators; some mathematicians regard them as the only 'scientific' results in mathematics education. Indeed, these results provide information which frequently seems 'objective', 'scientific' and easily intelligible. Apart from the scarce scientific relevance of some of these results, for the reasons given above, we have seen in our research how the *quantitative evaluation* of pupils, teachers, school systems, projects and innovations may cause serious damage (for instance, it may orient teachers towards preparing pupils to be successful in quantitative assessment tests - an aim that can conflict with importasnt long term educational goals).

Other reasons for the unreliability of ME experimental research results stem from some characteristics that ME shares with human sciences - for instance:

- the importance of (and the difficulty of controlling) ideological assumptions. Sometimes a teaching innovation is tested, or a research hypothesis is chosen, in accordance with the researcher's ideological positions. This can have consequences both on the acceptance and diffusion of research results and on the processes for testing research hypotheses;

- the danger of arbitrary extrapolations from one cultural setting to another. As an example, we may cite (from Boero & Szendrei, 1998) the differences in Italy and in Hungary concerning the learning of natural numbers: in the Italian language, natural numbers *('one, two, three, four...')* are used to indicate the days of the month (the sole exception being the first day, commonly named *'the first of...')* ; by contrast, in Hungarian, all the days of the month are named with the ordinal adjective *'the first..., second..., third...'*.So in the two countries the relationships between 'cardinal' and 'ordinal' aspects of natural numbers differ in pupils' early mathematical experiences. These differences may affect both the cognitive hierarchy between different aspects of the number concept and the opportunities that real life situations offer the teacher for approaching these aspects in the classroom, with heavy consequences on learning results.

3. How to improve the reliability of ME experimental research results

In my opinion, ME can improve the reliability of presently produced results, but the price to be paid is not low. Specifically, when reporting experimental ME results clear and constant reference should be made to the specific conditions in which they were produced (including the students' cultural setting and preceding school experience). This implies strong a priori limitations on the validity of results, but seems necessary in order to avoid dangerous extrapolations and arbitrary applications.

On the other hand, researchers (and teachers) should try to keep methodologies and results of the 'quantitative information' type under control. For instance, it is crucial to understand whether it is possible (and, in this case, how it is possible) to keep mathematics education variables 'constant' in order to create effective control groups.

Another necessary constraint comes on the mathematics side. In recent years (cfr. NCTM 1989 standards) some crucial parts of mathematics (like "proof" and "algebraic language") have been made less important in assessment, or even removed from the basic requirements of high school in some countries. Action of this kind should be carefully taken under ME control and, if necessary, discouraged with scientific arguments. Indeed not only does it provoke well justified reactions from mathematicians, parents, and others (see "Math Wars" in USA), but is also dangerous for a number of reasons:

- it may compromise the quality of the students' intellectual development (especially if we allow that the content of the activity is relevant for it: see Vygotsky, *Thought and Language*, Chapter VI, about the role of grammar - and algebra - in the students' intellectual development);

- it can compromise the specific role of cultural transmission that schooling plays within the mechanism of cultural reproduction in a society.

The fact that "proof" or "algebraic language" (as presently taught) are difficult for most students is not a good reason for cutting or delaying them in curricula. ME must accept the challenge coming from this kind of topics. Some, preliminary ME results show the feasibility of alternative, successful approach to them. But promising results need a strong commitment both on the historical -

epistemological ground and on the cognitive ground, and a sound mathematical perspective (as an example concerning "proof", see Mariotti et al. 1997).

As a consequence, ME teams need to include researchers with a sound mathematical and epistemological grounding. The reasons for their inclusion lie not just in the need to avoid relevant mathematics topics and skills being omitted from or misinterpreted in ME; they can play other, significant roles. For instance, they can help detect possible connections between students' strategies and crucial mathematical behaviours; they can establish possible epistemological links between different investigations or research hypotheses, etc., thus widening and strengthening the scope of ME.

In my opinion, another necessary component in ME experimental research is the presence of teacher-researchers. From a purely academic-centred perspective, researchers are expected to be professionals operating in the field of research with an academic status. But what about research questions coming from school practice and the application of research results in the school system?

In the Italian ME tradition (see Arzarello & Bartolini Bussi, 1998), the participation of school teachers in all the phases of experimental research (with full researcher status, including the right to take part in decision making about the research questions, methodology, etc. and in analysis of experimental data) would provide a number of advantages: a guarantee that research questions are strictly linked to acute difficulties faced in school; the exploitation of teachers' awareness about the research methodology and hypotheses in managing teaching experiments is exploited; personal knowledge of students (when analysing their performance); a better choice of results to be diffused in the school system, and their optimum presentation (for instance in pre-service and in-service teacher training: cf Boero, Dapueto & Parenti, 1996).

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DOES RESEARCH MAKE A CONTRIBUTION TO TEACHING AND LEARNING IN SCHOOL MATHEMATICS? REFLECTIONS ON AN ARTICLE FROM DIANE RAVITCH

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Abstract: This response to Diane Ravitch accounts for some differences between medicine and mathematics education, but accepts some of the criticisms as valid.

EVIDENCE-BASED PRACTICE

The article powerfully contrasts the position of the knowledge-base in medicine with that in education. This is in line with current political thinking in the UK where it is also now fashionable to contrast research in education unfavourably with medical research. The result is to justify a call for a move in education towards 'evidence-based practice'.

'Evidence-based practice' has been adopted as an aim in medicine and nursing. The Cochrane Collaboration in Britain sponsors groups of researchers and practitioners in various healthcare specialisms to carry out systematic reviews of existing research. The studies included must meet certain conditions of rigour, if possible employing randomly controlled trials. The findings in these reviews are used as a basis for recommending current best practice in healthcare. Our UK Government Department for Education and Employment has recently funded a Centre for Evidence-Informed Policy and Practice to extend these reviews into education.

My personal position is that I cautiously support this move into evidence-based practice but am sceptical about its appropriateness at present due to factors which I discuss below. It should also be noted that there are important areas of research in mathematics education (e.g.epistemology, policy studies) to which this discussion is irrelevant.

1. The robustness of the scientific knowledge-base

Both medicine and education are eclectic fields in that they are vocational areas which use a knowledge-base drawn from more fundamental disciplines.

In the case of *clinical medicine* the knowledge-base is mainly biological and biochemical and is reasonably well-established in some areas. For example inrelation to pulmonary embolism, which is the subject of this article, one can point to several centuries of scientific research into how the blood vessels, heart and lungs function. Some of the scientific knowledge may have been gained in the medical context by doctors or researchers, or it may arise in the laboratory.

Of course Professor Ravitch was a bit lucky that the medical field in which she was suffering was one that was relatively well-understood. If she had been afflicted by certain types of cancer, schizophrenia, obesity or multiple sclerosis she might have been less impressed by the competence of the medical fraternity, since the biological bases of these diseases are less well understood.

Education rests on a much less well-researched knowledge base in terms of fundamental disciplines. We are at last gradually beginning to gain some insight into brain functioning via tomography, but psychology is a new and as yet very empirical science, with many unconnected detailed results and a few theories, which are too general to act as a reliable guide for practice. I am old-fashioned enough still to believe that psychology, both cognitive and social, is the key discipline in teaching and learning, but those parts of sociology and other disciplines which relate to education are not much further developed.

In relation to mathematics education in particular we have very little idea about the nature of schemas and how they relate to classroom events. We do not know in what form knowledge is internalised, assimilated or accommodated, how it is associated or conceptually related to other knowledge, and especially to generalisations, abstractions, specific language or symbols or diagrams. We do not know why some pupils seem to acquire and re-structure ideas quickly, nor why some have more facility than others at spatial representation, why some are more curious about mathmatics or why some are more able to make creative leaps. Nor do we know much about the effect on learning of different aspects of teachers, teaching or teaching materials Thus it is not surprising that most research in mathematics education is at the stage of 'nature study' i.e. careful observation and local theory, in order to build up fragments of the knowledge-base.

2. The evidence-base for effective practice

Distinct from but connected with the knowledge-base is the applied research which evaluates practice. It is obvious that a fuller theoretical understanding is likely to lead to more effective practice. Equally, evaluation results should lead to further investigation and thus further knowledge. There has been well over a century of medical practice which has been based on at least some elementary but robust scientific knowledge.

Thanks to the professional bodies in medicine, innovative treatments reflecting greater understanding (and sometimes lucky hunch) have first been reported in the medical literature as interesting case-studies, and later where appropriate tested against other treatments in larger scale trials, or evaluated post-hoc using data-sets. Thus there is a tradition of what we might regard as informed practitioner action-research followed up by larger scale quantitative studies.

In the most advanced areas of medicine, such as pulmonary embolism, these procedures are highly effective. However it should be noted that in other areas such as Alzheimer's syndrome or lung cancer this state has not been reached, whether because of insufficient understanding of the biological condition or because of insufficient research on possible preventative measures or cures.

This method of informed action-research followed by more systematic evaluation of potentially promising practice would seem to be a reasonable way to proceed in education, and of course some start has already been made.

To take areas in which I have been involved personally, a 'pure' research project identifying cognitive hierarchies in the mid -1970s, Concepts in Secondary Mathematics and Science, formed the main knowledge-base for several developments carried out by or in co-operation with teachers. These include a published curriculum scheme (SMP 11-16), two published formative assessment scheme (Graded Assessment in Mathematics/Science - GAIM/GASP) leading to the original 10-level form of the English National Curriculum, and two published schemes of occasional 'thinking' lessons aiming at cognitive acceleration in mathematics/science education (CAME/CASE). All were developed formatively through small-scale and then wider-scale trialling and modification, but only the latter schemes have had a formal evaluation in terms of matched controls and effect-sizes. Nevertheless there have been other quantitative but less rigorous data which have suggested success, like 65% market penetration for the SMP scheme and national test data which mainly validated the content of national curriculum levels.

There are of course ethical problems in trialling educational materials and practices, but some of these are similar to those in medicine. For example the confidence and often the formal permission of the patient/doctor/hospital or the teacher/parent/pupil/ headteacher may have to be gained. Nevertheless this might well be easier for a prestigious medical institution recommending an apparently miraculous and possibly quickly administered new treatment than for a lower status educational researcher who needs permission for whole classes of children for a change of curriculum or teaching practice lasting several months.

While larger scale medical experiments seem to be able to standardly use random allocation of patients to treatments, it is more difficult to envisage this being possible in education, even at the class or school level, although there are some powerful examples, e.g. in Bob Slavin's work. It is certainly both easier and cheaper, although less rigorous, to look for well-designed experiments with matched pairings. The differences in tradition are likely to be partly due to more generous research funding in medicine but partly also because in most educational experiments the commitment of the teacher is crucial. This may be similar for surgical procedures but is presumably less so if the main treatment is medical in character.

I believe that we should aim for more rigorous experimental design in mathematics education; we should be prepared to submit our favoured solutions to a fair test and consequently need to be more ambitious in our funding requests. The result would be more credibility with government; it is currently difficult to persuade them that any contribution is made by our work to raising standards in mathematics learning. Unless we act, changes may be taken up by teachers but may not deliver better results, and more worryingly sound recommendations may not be adopted because people are not convinced that the benefits will be worth the investment.

3.The complexity of educational contexts

Professor Ravitch's main point is that where educationists are constantly bickering about validity issues, medics tend to get on with it in more robust ways. There are of course more and less controversial areas of medicine; for example the arguments rage about homeopathic versus intrusive treatments for some forms of cancer, or therapeutic versus drug treatments for depressive and behaviour disorders. Whereas some diagnostic tests may be well-accepted, there are almost certain to be others which are just as much argued over as in education. As noted earlier, problems are more likely to arise in areas with more contested knowledge-bases. And it should be noted that teachers like doctors are likely to be robust practitioners while in both areas the academics argue and may sadly sometimes be more concerned with academic status relating to theoretical niceties than impact on the system. One interesting difference is that in medicine the academic is more likely to be also a practitioner.

Educationists can argue that things are more starightforward in medicine; popping pills is less complex than teaching in a complex social context, and a physical measurement of e.g. blood pressure is simpler than an assessment of mathematical learning or attitude. It is certainly true that in education, significant effects in properly designed experiments are difficult to come by, perhaps because of these multiple interfering factors. But there must be many areas of medicine where the patient's history, attitude, social relations and background are also significant factors in the success or failure of a treatment. Many of these variations are ironed out by randomly controlled trials given a sample which is large enough, which relates back to the previous point.

CONCLUSION

In this response I have tried to make it clear that the contrast between medicine and education is less great than is presented and can be partly accounted for by the different stages reached both in the development of knowledge and in evaluation research. There are probably other contributing factors, like the fact that it is easier in education to build up a long publication list through critique and reflection, based on a little observation, than by rigorous large-scale experimentation. Or that a small improvement in medical treatment may still be worthwhile achieving if one or two lives are saved, and may be easy enough to disseminate to a small number of well-networked practitioners. Even a few percentage points improvement in international comparisons of mathematics achievement may seem little reward for the expense and disruption of systemwide change in mathematics teaching.

Nevertheless I believe the article is useful in calling us to account; if we are not in a stronger position to provide evidence of our successful impact on mathematics learning in 50 years time, then it will be difficult to find suitable excuses.

BEST COPY AVAILABLE

An Approach for Developing Well-tested, Validated Research of Mathematics Learning and Teaching

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1. Investigating Local Learning Issues from An International Perspective

Education research from international perspectives should aim to reveal educational issues in the society, and to develop appropriate strategies to resolve these issues under international collaboration.

A comparison study (Lin, 1988, 1991; Hart, 1981, 1984) that focused on the societal differences of reasons why Taiwanese and English secondary students develop their misconceptions / incorrect strategies of solving mathematics problems have shown its values on investigating local learning issues from an international perspective. I will argue that the approach of investigating local learning issues from an international perspective do generate a methodology that makes well-tested, validated education research meaningful.

2. English "Address" vs. Taiwanese "Integer-Multipliers"

Among the populations of age 13~14, there are about 30% and 25% of English and Taiwanese students respectively are using the incorrect-addition strategy consistently on "hard" ratio tasks (Hart, 1984; Lin, 1991). That is, those students in both societies have the same problem solving behavior on certain ratio tasks with non-integer rate. When investigating the reasons why those students reason in such a way, Hart (1984) called those English students as "adders" for the characteristics of those English students are identified as follows:

a) using addition-based child-methods, such as halving, doubling, adding on and building up to solve easy ratio items;
b) avoiding application of multiplication of fractions, and taught algorithms; and
c) never using multiplicative strategies, such as the unitary method (how much for one), and the formula method (a/b = c/d) (Hart, 1984).

However, we found that those counterparts of the Taiwanese are able to use multiplicative methods such as multipliers and unitary methods predominantly on easy ratio problems, and switch to the incorrect addition strategy on hard ratio problems. Clinical interviews with those students have shown that the main reason they fall-back to the incorrect strategy is because of non-awareness of non-integer multiples (Lin, 1991). For those Taiwanese students, their concepts of multiplication have strong linkage with their concepts of multiples. Since they don't recognize non-integer multiples as multiple, when they faced with ratio tasks with non-integer rate, harder than $\frac{n}{2}$, naturally the linkage between the tasks and the multiplicative strategy can't be connected in their mind. The incorrect addition strategy seems then to be the reasonable choice for them. We called those students, about one quarter in Taiwanese population of age 13~14, the "integer-multipliers".

3. Bridging the Prior-Framework to the Formal Conception

The aim of investigating learning issues is for educators to re-design more effective mathematics teaching, to enhance effective learning in classrooms. Bridging one's prior-framework to the formal concepts is one view of learning. The prior-framework is used here to be a general terminology which could be called as misconception, pre-conception, child method, informal knowledge, intuition, alternative framework, spontaneous concept, etc., within specific contexts in the literature. Under this view, effective learning can be interpreted as the involvement of learners in developing connections between their prior-framework and the formal concepts.

Following the view of learning as bridging the prior-framework to the formal knowledge and the findings of different characteristics of English adders and Taiwanese integer-multipliers, we examined and modified the diagnostic teaching module on ratio that was developed by Hart (1984) to help English adders. The version of Taiwanese diagnostic teaching module on ratio was proved to be an effective module for those Taiwanese integer-multipliers in developing their multiplicative strategy on proportional reasoning tasks (Kuo, et al., 1986). Furthermore, the finding about Taiwanese students do not aware non-integer multiples as multiple caused a great attention by Taiwan mathematics educators. Consequently, the new elementary mathematics curriculum implemented in 1993 did take this result into consideration and have developed activities for children to learn their concept on non-integer multiples. For instance, the topics on multiplication of fractions and decimals in this new curriculum are treated not only as procedural knowledge but more fundamentally as well as conceptual understanding in Year 5 and Year 6 respectively.

4. Making Sense of A Methodology

Some consequent statements have been drawn from the results of the above investigating.

a) Defining the position of research in mathematics education to be apply-oriented is meaningful. Results of researches were proved to be the key resources for re-designing teaching modules and learning activities, and reforming curriculum to meet the need of ensuring effective learning.

b) International comparative study could be aimed to formulate specific education issues within each participated populations and develop strategies collaboratively to resolve that issues.

c) The characteristics of learners' prior-framework in learning mathematics are societal and cultural rooted by nature.

Considering statements a), b) and c) as the basis of a methodology, I will exercise it on studying learning issues in mathematics arguments.

5. Investigating Learning Issues on Mathematics Arguments

In England, the Current Mathematics National Curriculum prescribes a process-oriented approach to "proving" that aims to make proof more accessible (DfE, 1994). Hoyles and Healy (1999) described that English "students' justifications are usually expected to draw on experimental results.

In Taiwan Junior High School, the Euclidean geometry, in simplified format, is still the dominant context teaching deductive reasoning. At this stage, algebra is not considered by majority of teachers aiming to play the role of proving.

Hoyles and Healy (1998) found out that the improper empirical methods are favored by about a quarter of English top 20% students of 15 years old as their own approach of proving the statements such as item A1: "When you add any 2 even numbers, your answer is always even", and item G1: When you ass the interior angles of any triangle, your answer is always 180°.

Considering the empirical approach as the main component of their prior-framework in learning with formal proof, then the learning path of those students can be represented as Figure 1. They are seeking for ways to transfer from the empirical approach to the formal proof.

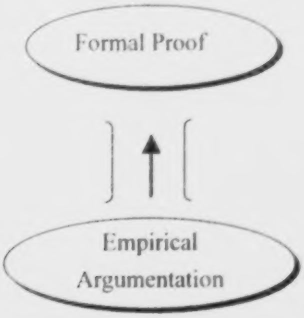

Figure 1. A representation of a learning path for English students

The formal-presented improper or incorrect arguments identified as their own methods by about 30%~40% of Taiwanese students. Considering those arguments as the main component of their prior-framework in learning with formal proof, then the learning path of those students can be represented as Figure 2. They are seeking for ways to modify their incorrect or improper formal argumentation to adapt the formal proof.

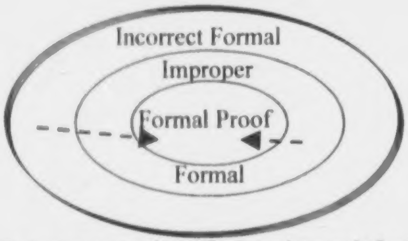

Figure 2. A representation of a learning path for Taiwanese students

Figure 1 and Figure 2 show that the prior-frameworks and hence the nature of issues of developing effective learning activities in mathematics proof are different between Taiwanese and English students.

6. A Conjecture of Linking the Informal Argument to the Formal Proof

Hoyles and Healy (1999) had tried to link informal argumentation with formal proof through computer- integrated teaching experiments. Hoyles reported that their teaching experiments did not seem to result in a very effective learning.

In Taiwan pre-pilot study (Lin & Chen, 1999), three either improper or incorrect formal presented argumentations on the item A1 shared a common feature that each one of those arguments start with naming one-unknown. To make a correct formal proof of the statement: The sum of any two even number is always even," students have to name two arbitrary unknowns actively, this assume to be over- loaded in their cognition at aged 15 years. Taking one- unknown as reference to establish the two even numbers is the common feature of those improper / incorrect formal arguments. Could this feature help us formulating a conjecture that the learning of proving this kind of statements, the number of unknowns which is necessarily to be given by learners in the tasks should start from one and then two and so on?

Though, there may have many ways to learn formal proof. Nevertheless, this observation is worthy to try out in the future study from both societies.

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RESEARCH FORUM

Theme:
Dynamic geometry

Coordinator: Colette Laborde

Presentation1: Ana Paula Jahn
New tools, new attitudes to knowledge: The case of geometric loci and transformations in Dynamic Geometry Environment

Presentaion2: Lulu Healy
Identifying and explaining geometrical relationship: Interactions with robust and soft Cabri constructions

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New tools, new attitudes to knowledge: the case of geometric loci and transformations in Dynamic Geometry Environment

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The introduction of computer environments brings particular features into the didactic system: a computer (with convenient software) becomes part of the "milieu" and is the scene of the pupil's mathematical activity. It is of particular importance to determine the characteristics of "milieux" integrating a computer environment, and as far as we are concerned, the effects of the use of Cabri-géomètre in specific situations in order to analyse observed behaviours. Broadly speaking, we know that a computer provides new tools to act on abstract objects, in particular mathematical objects, and in this way it changes the attitudes of the pupil to the objects themselves.

We propose to discuss the fact that the computerised "milieu" with Cabri-géomètre opens new possibilities of action and provides an extended feedback: graphical and computational possibilities offered by this microworld allow a "reification" of geometric objects, as well as of numerous operations on these objects and various feedback. Furthermore, this "milieu" provides specific representations of knowledge related to the functionalities of the computer environment - fast treatment of figure deformations, dynamic apprehension of curve generation, specific tools such as "Locus" and "Trace" (particularly efficient in representations of images by a geometric construction, we will discuss it farther), fast and handy construction tools, such as "Parallel line", "Midpoint", etc. Indeed, the cost of operations, which is usually prohibitive in the paper/pencil environment, is changed due to these tools, and the economical constructions they afford make repetitions possible.

We are therefore interested in the pieces of knowledge that are related to computer environments, i.e., in their nature, as well as in the knowledge they are likely to bring into play in the apprehension by the pupils. We will look more particularly at the notions of geometric locus and transformation, through an analysis of the tools provided by Cabri: "Locus", "Trace", and those that represent transformations - "Reflection", "Symmetry", "Rotation", "Translation", "Dilation".

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On the notion of geometric locus: relating geometric and functional aspects

Traditionally, in the context of synthetic and static geometry, a geometric locus is likely to be perceived as a set of points satisfying a certain property, the set being regarded either globally, or point by point.

"*In order to say that the points of a figure satisfy the same property, and that all points satisfying this property belong to the figure, we say that the figure is the geometric locus, or the set of points satisfying the mentioned property.*"

Therefore, a geometric locus is a **set of points** having a certain property: a characteristic condition determines whether a point belongs to the set.

Loci are not characterized in the functional setting, or in the setting of transformations, they serve as means of transition between the definition of a figure in the Euclidean sense (taken globally), and its definition as a set of points. In this way, the term "locus" can be replaced by "set of points". That is the case, for instance, of "classic" pointwise characterisations of objects, such as perpendicular bisector, angle bisector, conics, and more particularly circle.

The tool "Locus" in Cabri does not come under the same meaning. Indeed, it is not possible in Cabri-géomètre to obtain a locus from a metric property, or a property which is not expressed in a functional form, i.e., as a condition relating a point with only one degree of freedom (i.e. point on object) to another, variable point[2].

In Cabri, a locus is a set L of points that are function of a point element of a set E:
L = {f(P), P in E}, and on the screen we have a sketch (a representation) of the set L by a finite number of f(P). The point P is a "variable" point that belongs to a certain set of points in the plane (a straight line, a circle, a line segment...), and the point f(P) of the locus is related to P by means of a geometric construction. As regards our purpose, this functional relation can represent a geometric transformation, and the tool "Locus" represents the means to obtain the set of images of points of the figure-object, i.e., the image of the latter.

With the "Locus" tool, the points that define the image are calculated by the software and obtained directly without the need for any movement of the variable point. The locus appears instantaneously and in its entirety as a new geometrical object on the screen. There is a second way that the user can locate the set of point-images of the figure under a particular transformation. With the "Trace" tool, the student can drag point P on any figure, and observe the trajectory of the point P', related to P by construction. The representation that is produced using "Trace", however, does not have the status of a true Cabri object, since it does not rely only on *geometric primitives* but also necessitates graphical movements. For this reason, this second way of exploring the set of points is not recognised as legitimate for mathematical study, it is conceived as no more than a visual aid.

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[2] It is possible using the Cabri "Locus" tool to construct loci of lines (envelopes), discussion in this paper is restricted to the loci of points.

In the following section, examples will be provided to illustrate how these two tools were integrated in a didactic sequence about geometric transformation. It was hypothesised that the these two tools, "Locus" and "Trace", would enable the clarification and description of what remains implicit in paper/pencil, especially the case of the functional relationship which is specific in their functioning. The sequence was designed for use with a class of 33 high school students (aged 15-16) from a school in southeast of France. The complete sequence consisted of four teaching situations realised during seven sessions, each of which lasted one hour. During their interactions with the activities proposed, students worked in pairs and five pairs were select for case-study. All of the students had had approximately 6-months experience of using the Cabri (both computer and calculator versions) as part of their mathematics class prior to taking part in the transformations study. In this paper, episodes drawn from the analysis of the strategies of the case-study pairs to the problems proposed in the third session will be presented to consider the contribution of the Cabri tools to the students developing ideas about geometrical relationships.

On geometric transformations...

Problems where transformations are to be used to prove a certain property, or where the transformation is not explicitly given, are rare in junior high school (11-14 years). From the beginning of high school (15 years), such problems become of major importance. Nevertheless, the transition from the *object* conception to the *tool* conception of a transformation, which is essential in the problem solving in geometry, is not made without difficulty. The activities from the didactic sequence[3] considered in this papers refer to the first of these conceptions, but before describing these in more detail, we identify briefly some ways in which Cabri offers new mediations of the tool conception.

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[3] For more details on a teaching sequence based on these choices, see Jahn (1998).

Transformations as tools

In Cabri-géomètre, transformations appear as ***construction tools***. The toolbox "Transformations" contains reflection, symmetry, translation, rotation, dilation, and inversion. These tools allow the construction of the image of a point and of other objects such as segment, circle, triangle, polygon, conic...

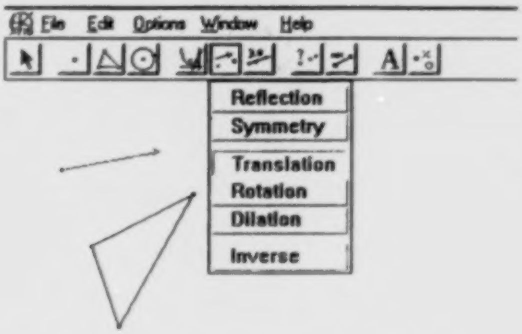
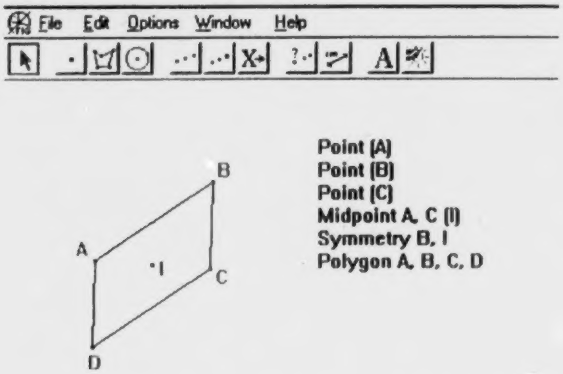


Figure 1 - Toolbox "Transformations"

Each of these tools is used in the following way: one first indicates (i.e., points out by means of the cursor) the object to be transformed, and then an element which defines the transformation (e.g., a point in the case of a symmetry, a vector in the case of a translation, a point and a number indicating the measure of an angle in the case of a rotation...). In this way, the image of the chosen object is directly obtained. By means of these operators, the transformations are approached from the global point of view: they act on objects. It is important to emphasise that these transformations are *geometric primitives*: they are, together with "Midpoint", "Perpendicular bisector", "Perpendicular line", etc., construction tools governed by geometric properties. This functionality is specific of the software and has no equivalent in paper/pencil environment, where the effective construction is unavoidable, unless one uses articulated systems (machines to draw mirror images, pantographs, etc.). Therefore we suppose that the fact that these tools are available in Cabri-géomètre may lead to a more systematic use of transformations in the construction tasks or in the study of a configuration.

Let us take an example of a parallelogram: given three points A, B, and C, construct with Cabri the fourth vertex D of a parallelogram ABCD. Its construction based on a symmetry would be a better sign of understanding than a statement such as "*a parallelogram has a centre of symmetry*". By the way, using Cabri, this construction is highly *economical* in the sense that only two tools are necessary, and the point D constructed in this way can be recognised for all positions of the three initial points, even when they are collinear.



Point [A]
Point [B]
Point [C]
Midpoint A, C [I]
Symmetry B, I
Polygon A, B, C, D

Figure 2 - Construction of a parallelogram

We can also consider the use of the tools from the Cabri toolbox "Transformations" to study possible procedures to construct a line parallel or perpendicular to a given line and passing through a given point (Capponi, 1993).

Transformations as objects

The first transformations taught in junior high school and high school are isometries, after which dilations are introduced; all preserve collinearity and angles, i.e., the shapes. At this level, it may seem evident to be satisfied with these applications: they are at same time simple and fundamental. However, one must be aware of the danger of a generalisation by the pupils who can believe, for instance, that the property "the image of a straight line is a straight line" is valid for all transformations. A systematic study of isometries may be the origin of a didactic obstacle according to which a transformation does not distort, it only displaces and turns objects. As a result, the relation between figure-object and figure-image is not established, and the notion of transformation is reduced to the idea that a figure can take up two different positions.

This is the reason why it seems interesting to study a few simple examples of transformations that do not preserve some property (collinearity, distances, ...) - they show its depth and non-obviousness. In other words, it is advisable to make students question the validity of theorems about preserving properties, and in this case, "deforming" transformations represent an unavoidable breach.

The use of a computer in such a kind of activity seems quite relevant, and using Cabri-géomètre is particularly convenient for this purpose. Let us mention two notable approaches:

- the characterisation of a transformation by means of its effects, especially by its invariants. In Cabri, a transformation can be presented as a ***black box*** that can be explored by moving around free elements of the figure and by experimenting with the *object,* using the Cabri tools. We call this the black box approach.

- the motivation of a **pointwise** conception of a transformation: that is to facilitate students from seeing geometric transformations as more than global operations on whole figures.

Black box approach
In this case, the starting point is not a textual definition of the transformation; rather, the definition and its characterisation are to be constructed by the pupil from the study of the effects of the black box on points or common figures, allowing to bring to the fore the invariants of the transformation. Clearly, this type of situation is impossible to carry out in paper/pencil environment as it is based essentially on displacement, as well as on geometric interpretation of spatio-graphical behaviours.

The following activity, taken from the third situation in our didactic sequence, provides an example of the black box approach. The students open a Cabri-menu configuration, in which a macro-construction simulating an oblique symmetry is available. This macro operates on points, that is it produces, on the basis of two given lines (a directrix d and axis a), the image of a point. It is not possible using this macro, or any other means, for the student to obtain directly the image of a entire figure. With the inclusion of this macro, it is possible in Cabri to execute a transformation, in principle unknown to the students and designated as X, in the absence of any explicit definition, leaving the students the task of characterising the transformation by studying the behaviour of a pair of points (P and P') and seeking to establish the properties by which they are related.

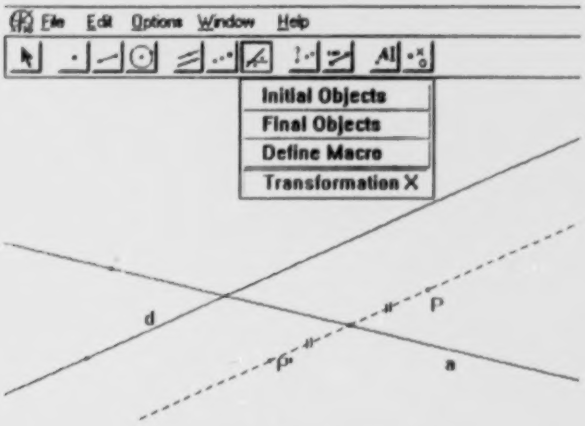


Figure 3 – Oblique symmetry

In the transformation approach, this type of situation was well appropriated by the students[4]; that is, its devolution was associated with positive results. Observations of the strategies of the five case-study pairs indicated that students' first concerns were limited to the consideration of perceptive relationships (for example, noting that the points were always on opposite sides of the line a, or coincident when dragged onto line a). The next step was to engage to explore the effect of different applying different Cabri-instruments to the figure (a segment or joining P and P', measurements of P and P' for the primitive lines etc). Through these manipulations the properties behind the macro construction became evident and all the students managed to construct their own point object image pair Q and Q' in which Transformation X was successfully modelled.

In relation to validation, it was sufficient for the students to see either that when Q was coincident with P, this was also the case for P' and Q' or that when the macro Transformation X was applied to point Q, that the resulting image-point coincided with the point Q' constructed by the students themselves.

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[4] In fact, students were already familiar with the black box approach having experimented with it before in relation to different geometrical content.

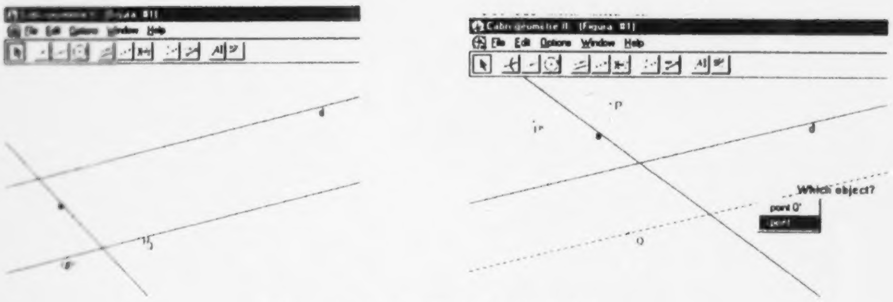

Figure 4: Two methods of Validation

The students also recognised that the constructed points, P' and Q' belong to the same transformation, despite their difference in terms of the Cabri-tools utilised in the process of their construction. This realisation was an important step towards the conceptualisation of a transformation as a mathematical object. For these students, a transformation is determined by a set of pairs of points, object-point/image point, independently of the manner through which the image point was constructed – or as one of the students, Ludovic explained, "*a transformation is a way of relating points...You have one point here* [he gestured with his hands] *and the transformation will connect it with another in some way*" This definition contrasts with those that emerged during an earlier discussion in which students had been asked to explain what a transformation was to them. Their responses indicated that the general term had little significance. Many students simply answered that they did not understand the question. Others were able to produce adequate explanations only if a particular type of transformation was specified (reflection, translation, etc), their initial reactions of the form *"...which transformation? Reflection?"* or *"...what do you mean? Is it to talk about a rotation for example?"*.

Pointwise conception

For the notion of transformation as a pointwise mapping to be consistent, which is necessary to achieve the high school objectives (or, in other words, if we want to help the students develop the dual global/pointwise aspect of a transformation which will facilitate the transition to the notion of pointwise mapping), the way of looking at figures must change: the **global** point of view has to give way to the **pointwise** point of view. This is far from natural for pupils at the beginning of high school. Our recent research (Jahn, 1998) shows that the pointwise approach to figures, and therefore to transformations, seems unthinkable in paper/pencil environment, it is not operationally practical. Yet, as we described above, it is possible in Cabri to define a transformation as a black box starting with the pointwise approach, i.e., by studying the procedure of construction of the image of a point and by giving its pointwise definition. Leaving isometries aside to avoid the possibility to determine beforehand the nature of the figure-image, the pointwise treatment is maintained and one can start studying the *figure F' formed from the set of points that are images of the points of the figure F*. The tool "Locus" available in Cabri appeals to a functional interpretation and favours the pointwise aspect: to use it, a figure must be seen as a set of points, and its image can be characterized as the set of points images of the points of the figure-object.

In the follow-up to the black box activity described above, for example, the students were asked to construct, with the help of Cabri, the image of a circle under Transformation X (oblique symmetry). This task characterised the passage from points to figures: how can the image of a circle be produced using the image of just one point if the properties defining the transformation are not known[5]? To consider this question, it was proposed to the students that they consider the images of 4 points of the circle-object and, through dragging, try to observe the behaviours of these points. The analysis of the images of the 4 points created the first favourable rupture: four pairs immediately began to doubt their initial idea that the image of a circle is always another circle of the same radius. This was the first time that they had observed a figure that was deformed under transformation.

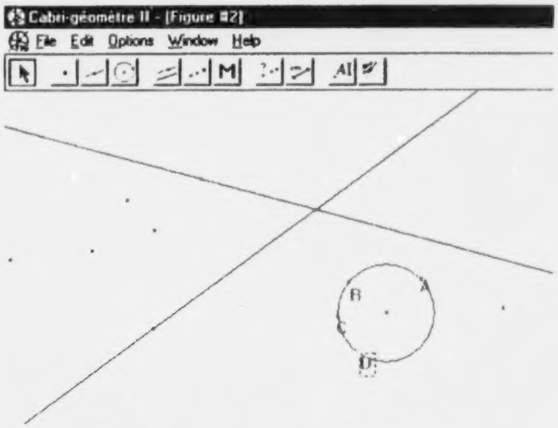
Cabri-géomètre II - [Figure #2]
File Edit Options Window Help
B
A
C
D

Figure 5: 4 image points of the transformation

The fifth pair held onto the idea that the image had to have the same form as the initial figure. Lil and Hortensia, tried to build a circle passing through the 4 image-

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[5] The resolution of the black box did not require an explicit analysis of the properties conserved by Transformation X.

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points, by constructing two perpendicular bisectors to determine the centre relative to three points and then observing that third perpendicular bisector did not pass though the same centre point.

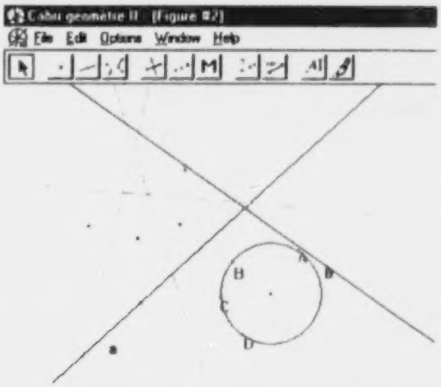
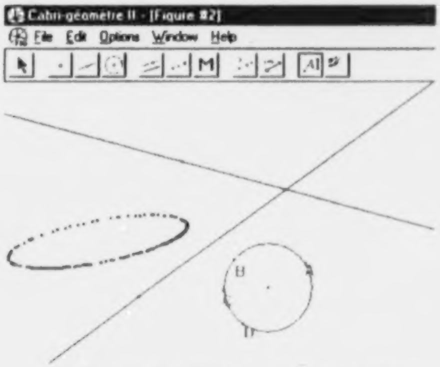

Cabri-géomètre II - [Figure #2]
File Edit Options Window Help
B
A
C
D

Figure 6: Trying to construct the circle

The rest of the students searched for Cabri-tools that would enable them to visualise the trajectories described by the constructed points. In their attempts to identify the form of the image-figure, they eventually made use of either the "Trace" tool (3 pairs) or "Locus" (1 pair). Lil and Hortensia also chose the "Trace" tool after their abortive attempts to construct the circle.


Cabri-géomètre II - [Figure #2]
File Edit Options Window Help
B
D

Figures 7: Resolving with "Trace"

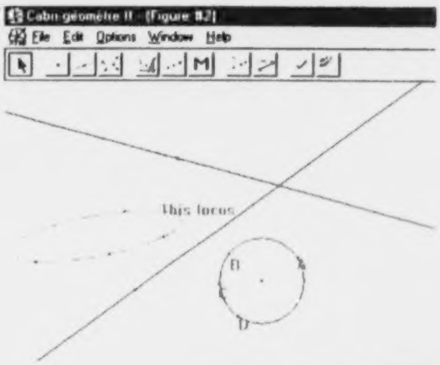

Cabri-géomètre II - [Figure #2]
File Edit Options Window Help
This locus
B
D

Figure 8: ...with "Locus"

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Passing through or arriving at Dynamic Understandings?

Even given the preliminary work with "Locus" and the inconveniences of "Trace" – the second situation was devoted to the simultaneous exploration of these tools – it was the second tool that the students preferred. Generally speaking, the students had great difficulties in understanding the order in which inputs should be selected to successfully apply the "Locus" tool. We suggest that the problems stem from the complex reasoning associated with the automatic and instantaneous apparition of this object on the screen, prioritising a static view of the notion of variable and necessitating an initial input that is a member of another geometrical object.

The "Trace" tool provides a dynamic solution through which students can begin to negotiate this complexity. The physical actions upon the variable point, under the control of the student, facilitate them in constructing mathematically consistent meanings which can be transferred to the more opaque functioning of the "Locus" tool. The passage to the "Locus" tool emerged principally with the need to save the figure, or because it is impossible to move the trace-trajectory (technical characteristics of the software system). In fact, though important if it is to be acted on, or with, in further constructions, the production of a (quasi) true geometrical object was not strictly necessary for the resolution of any of the problems presented in the didactic sequence.

Perhaps this is why the advantages of the inclusion into the learning milieux of the dynamically-orientated "Trace" tool were more evident. In the beginning of the study, this tool was viewed as an accessible starting point from which the ideas behind its more static counterpart could be introduced. This view changed over the course of the experiment as it became clear that the "Trace" tool helped students to build meanings for a variety of mathematical notions – figure as set of points, the transformation as a functional relationship between input and output points, variable in geometry. It also supports the development of a new set of heuristic strategies, which, when actualised in the innovative problem situations – like the black box approaches – also afforded by the dynamic environment of Cabri, enrich the exploration of spatio-graphical phenomena in way which signal their possible geometrical interpretations.

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Identifying and explaining geometrical relationship:
Interactions with robust and soft Cabri constructions

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Abstract

This paper describes our attempts to introduce notions of proofs in Euclidean Geometry using the software, Cabri- Géomètre. The idea behind the teaching situations was that they would support students in switching between inductive and deductive concerns as they constructed various geometrical properties and justified the relationships between them. The paper considers two different kinds of Cabri-construction that were built and investigated by students as they interacted with this dynamic geometry system. It goes on to present an analysis of some of the ways in which, during students' early interactions with the deductive system of Euclidean proof, the interplay between empirical and theoretical modalities and the aspects of the proving process that are investigated can vary according to the type of Cabri objects constructed by the student.

The process of constructing a geometry proof is clearly a complex one. It involves both an appreciation that certain geometrical facts emerge as a consequence of certain others and the organisation of a coherent sequence of transformations by which the second set of properties can be inferred from the first. If the second of these factors, the deductive-axiomatic form, is introduced to learners in the absence of connection to any empirical reference, all the indications are that proof will be viewed as an inaccessible and meaningless ritual involving memorisation and reproduction (see, for example, Harel and Sowder, 1998). On the other hand, if attention to deductive reasoning is delayed and proof introduced in the context of empirical experimentation, students seem to understand better what is required of a proof, but are unable to construct one (Healy and Hoyles, 1998). A third alternative is to search for learning contexts which help students switch naturally between deductive and inductive concerns – contexts in which it makes sense to formulate statements and definitions through agreed procedures of deduction without severing any connection from empirical justification.

Dynamic geometry systems like Cabri-Géomètre have several features to suggest they might offer such a new context including:

- the construction and creation tools through which students can produce a diagram that is simultaneously a *drawing* and a *figure* (Laborde, 1993);
- the dragging tools which enable students to examine their constructions, both to identify relationships which remain invariant and to impose further relationships visually (Hölzl, 1996; Arazello, Micheletti, Olivero and Robutti, 1998);
- the trace/locus tools that can be used to produce representations of set of points that satisfy a particular property (Jahn, 1999);
- the tools to re-examine and re-play construction procedures and offer students access to a language of descriptions and communication (Healy and Hoyles, forthcoming);
- check-property tools which allow the students to consider the domain of validity of visually identifiable properties of their constructions (Laborde and Laborde 1995);
- and measuring tools which permit students to consider particular cases and provide a different means of focussing on invariant relationships (Healy, 2000).

While Cabri- Géomètre offers the learner a model of Euclidean model with which they can experiment, as Laborde and Laborde (1995) point out, it is quite possible that the powerful tools of the system will legitimise empirical activity rather than encouraging constructions and explanations motivated by the mobilisation of geometrical knowledge. In terms the teaching and learning of proof this raises an important question: Can we come up with new types of activities harnessing the potential of the Cabri microworld to (a) encourage students to focus on the relationships between geometrical objects and (b) provide the means for students to develop arguments to explain why these relationships hold? This paper will outline our attempts to devise activities[1] which could serve as an introduction to Euclidean Geometry as a deductive system. The activities were created for use with for mathematics students (aged 14-15 years) of above average attainment in our country. It is important to stress that our expectation was that the students who worked on the tasks we devised would be novices in two respects. The software was new to them and, since this aspect of geometry does not have a high profile our current mathematics curriculum, so were formal Euclidean geometry proofs[2].

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[1] The work reported in this paper was carried out with Celia Hoyles during the project *Justifying and Proving in School Mathematics*, funded by the Economic and Social Research Council (Grant No. R000236178). I wish to acknowledge her central contribution to all aspects of the work reported here.

[2] The term "formal proof" is intended to carry the same sense as suggested by Grenier (2000), to represent a argument presented as a succession of sentences, using conventional mathematical formulations, showing how starting from assumptions(regarded as true) one can arrive logically to a conclusion.

Designing the activities

A critical feature of the learning activities we were aiming to create was that they would support an interplay between examples and their mathematical structure which would permeate all aspects of students activity. We wanted students to attend *explicitly* to the properties of the data they created from the outset – during their construction rather than only during their manipulation. Our view is that this is more likely to occur in contexts where students actions are accompanied by a corresponding mathematical description. An example may help clarify why.

The process of constructing a Cabri-square[3] is quite different from the process of producing a square using paper-and-pencil. When drawing a paper-and-pencil square it is not *necessary* to be mindful of the geometric properties that define its construction. A square-like sketch can be produced – without actually constructing any geometric properties – and afterwards it is possible to signal various properties indicating its squareness (see Figure 1). That is, the action of drawing a square can be quite separate from the expression of its properties. In this scenario, all the properties of the square arrive simultaneously as a block, there is little sense of any relationship between the properties or, more particularly, that certain properties of the construction emerge as a consequence of certain others.

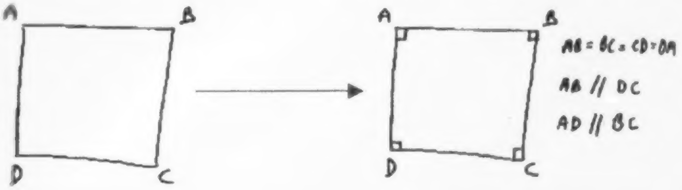

Figure 1: Sketching a paper and pencil square then signalling some of its properties

In contrast, the construction of a Cabri-square is the result of a process in which the user – *necessarily* – makes its definition explicit (Laborde, 1993). Students' action and the expression of mathematical properties are not disjointed or organised in a linear fashion (first action, then expression). Instead they can be one and the same. In this way the beginnings of a proof, the identification of the "given properties", is embedded into the students' activity – it is not something tagged on at the end, or provided from "above", but forms part and parcel of the mathematical model by which the student chooses to represent the problem situation[4]. Furthermore, once constructed (through a series of mouse clicks and menu selections) the user can re-enact the process of construction and re-access the definition of the givens.

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[3] "Cabri-square" is used to mean a robust construction which retains the properties of a square (cannot be messed-up) regardless of which of its components are manipulated on the screen.

In the Cabri context, it makes sense to investigate relationships between the various properties of a square by organising them in terms of given properties and those to be deduced since this reflects its construction. This is an important step into the proving process but it is not yet enough. The constructed objects need to be manipulated, in ways that not only help to identify these further properties, but also to clarify the transformations, the intermediary steps, by which these further properties can be inferred from those used by the student as the givens. The software may have a role in both these processes too, with the various means available to assist in formulating and verifying conjectures also offering different methods through which the steps to their proof may be made more visible.

Hence, our vision was one in which students employ the Cabri tools construct, manipulate and check geometrical relationships, receiving computer feedback brought about by their activities that could help towards the proof of any conjectures they formulated. To this end we devised a sequence of activities, each of which included a computer component with a common structure: students were to construct mathematical objects on the computer, identify and describe the properties and relations that underpinned their constructions, use the computer resources to generate and test conjectures about further properties, and make explanations as to why they must hold. The computer work was accompanied by paper-and-pencil exercises and by teaching episodes in which new mathematical ideas or additional software tools were introduced by the researchers.

Were students able to use the various facility of the Cabri software to successfully negotiate the activities as we intended? Was it the case that they became more cognisant of the geometrical properties and relationships of the visual artefacts they built and manipulated? Did their interactions which these artefacts help them understand why and when these relationships exist (or not)? In the remainder of the paper, examples from the work of two different students pairs, Tim and Richard and Karen and Abby, will be presented. These examples are not meant to be exhaustive[5], but have been chosen to consider two different methods that were be used to construct Cabri objects and how each method can be associated with different ways of traversing between empirical and theoretical modalities and with explorations of different aspects of the proving process.

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[4] Note that these ideas are not limited to Cabri-Géomètre but can be applied more generally to the analysis of student activity in microworlds for mathematical learning – for an elaboration see Noss and Hoyles (1996).

[5] The overall results of both the geometry and algebra teaching experiments are described in Hoyles and Healy (1999).

Experiencing geometrical dependency

In common with other researchers (see, for example, Mariotti, 1997; Gravina, 2000), when we began, it was our intention to encourage students to build *robust* constructions like the Cabri-square described above. In practice, we found that some students preferred to investigate a second type of Cabri-object, *soft* constructions, in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student. An examination of the different ways our two student pairs approached the first set of activities might help clarify the differences between these two construction types.

The idea of this activity-set was that students would explore various methods of constructing a second triangle using different combinations of the properties (sides and angles) of an existing (and general) Cabri-triangle with the eventual aim of identifying which conditions were sufficient to ensure congruency. In this activity were we not expecting students to construct any formal proofs, but we did want them to experience how the construction of some properties necessarily (or not) results in other geometrical by-products.

Before starting on this investigation, students were introduced to, and experimented with, a small sub-set of the Cabri tools for creation, construction, manipulation and verification of geometrical objects. These included (amongst others) the two constructions tools – macros – which we had added to the Cabri construction menu specifically for this task. The first, compass[6], allowed them to construct sides for their triangle equal in length to that of the original. The second, angle-carry, enabled the construction of congruent angles. So how did the students employ these tools during the activity? Tim and Richard built robust constructions. Karen and Abby preferred to work with soft constructions. Let us begin with the robust approach.

The robust approach

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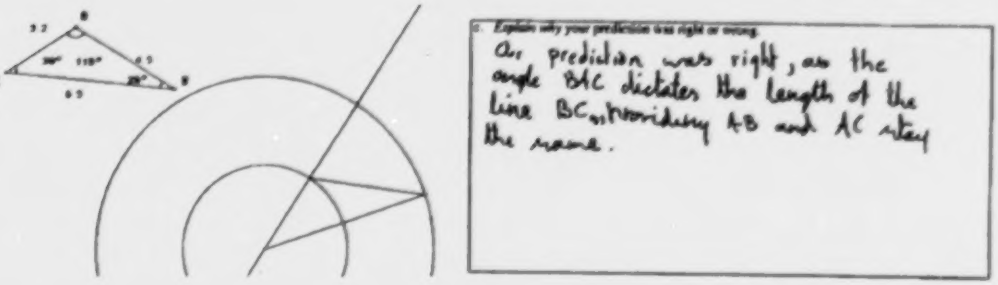
[6] These activities were all designed for Cabri-Géomètre è I, hence the compass construction was not one already provided in the default menu configurations as it is in Cabri-Géomètre II. The rationale behind the choice of the older version is explained in Healy and Hoyles (forthcoming).

To investigate each condition for congruency, Tim and Richard constructed a second triangle so that it shared the appropriate three properties with the first. For example, when exploring the condition Side-Angle-Side – which they predicted would ensure congruency – they started by employing compass to construct sides equal in length to CA and CB, created a line segment of length CB and then constructed an angle congruent to $B\hat{C}A$ using angle-carry. The point of intersection where this constructed line crossed the circle of radius CA was used as the third point of their triangle. Their immediate reaction was that the constructed triangle (shown in Figure 2) was congruent to the first.



c. Explain why your prediction was right or wrong.

Our prediction was right, as the angle BAC dictates the length of the line BC, providing AB and AC stay the same.

Figure 2: A robust construction of triangle with the properties Side-Angle-Side of triangle ABC along with Richard's attempt to explain the constructed triangle must be congruent to ABC.

The pair went on to discover that it was possible to drag two of the three vertices of the constructed triangle, the point on the circumference of the larger circle, in which case the triangle was rotated, or the centre point of the circles, which had the effect of translating the triangle. In both these transformations, the dimensions of the triangles are not changed, hence it remained congruent under dragging. Finally the boys altered the measures of the original triangle. The three sides of their constructed triangle side changed accordingly and the pair concluded their original prediction had been correct.

Two observations can be made about Richard's attempt to explain why their constructed triangle was necessarily congruent to the original (presented in Figure 2). First, he appears to be thinking in general terms – at least in the sense that he makes no reference to the specific measures of the triangles with which he and Tim worked. Second, the language he used suggests some grasp of idea of the geometrical dependency that we were hoping the activities would foster – the use of the word "dictates" indicates a sense that the property of the third side of the triangle emerges as a consequence of the three constructed properties.

The two boys approached the condition Side-Side-Angle in exactly the same way, with both predicting again this combination of properties sufficient for congruency. To build their robust construction, they used the compass tool to copy the lengths CA and CB from the original triangle and then employed the angle-carry option, this time constructing a congruent angle corresponding to $\hat{CBA}$. The line resulting from application of this tool (in Figure 3, this is labelled r) crossed the circle of radius CA twice.

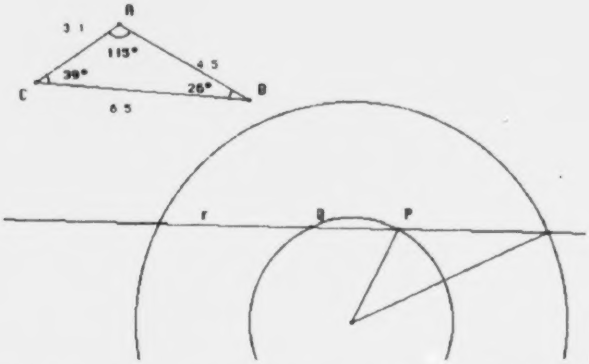
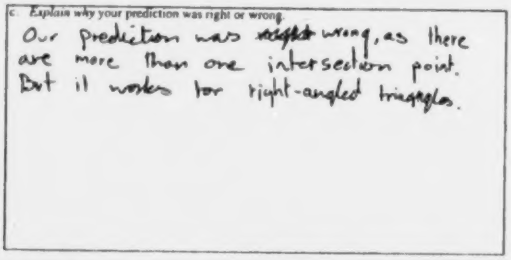


Figure 3: A robust triangle constructed using only one of two possible intersection points

At first, neither Tim nor Richard noticed the second point but, as they explored the figure, using the characteristically radical dragging movements students tend to use when exploring a robust figure[7] that they believe cannot be messed up, they eventually became aware of the second point. In this way, they saw that it was possible to create both a triangle that is *and* a triangle that is not congruent to the original one. Tim and Richard's investigations did not stop with this discovery, instead they went on to attempt to uncover the conditions under which the second intersection point disappears, uncovering one particular set of triangles, right-angled ones, for which the Side-Side-Angle condition does guarantee congruency (Figure 4).



c. *Explain why your prediction was right or wrong.*

Our prediction was ~~right~~ wrong, as there are more than one intersection point. But it works for right-angled triangles.

Figure 4: Richard suggests a set of triangles for which Side-Side-Angle guarantees congruency

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[7] Perhaps this is rather like using very large numbers to test an algebraic generalisation.

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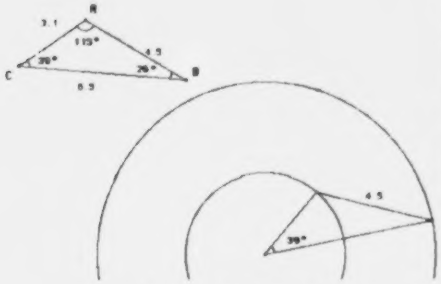
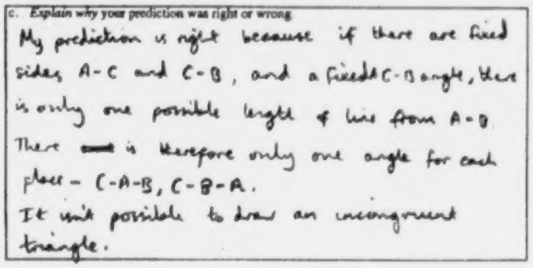
The soft approach

Karen and Abby approached the investigation in a different way. Instead of constructing all three properties they constructed only two, using the compass tool to construct sides of length CA and CB. Next they marked and measured the angle of interest – the angle corresponding to $B\hat{C}A$ for the Side-Angle-Side condition and $C\hat{B}A$ when exploring Side-Side-Angle. Their strategy involved carefully dragging the point on the inner circle until the angle measure matched that of the original triangle ABC. In the case of Side-Angle-Side, Karen and Abby found that there were two possible locations on the circumference of the inner circle for which the required angle of 39° was obtained. Measurement of the one side that was not constructed under their control confirmed that resulting triangle was congruent (Figure 5 illustrates one of the two "correct" locations and how Abby explained that her predictions was correct).



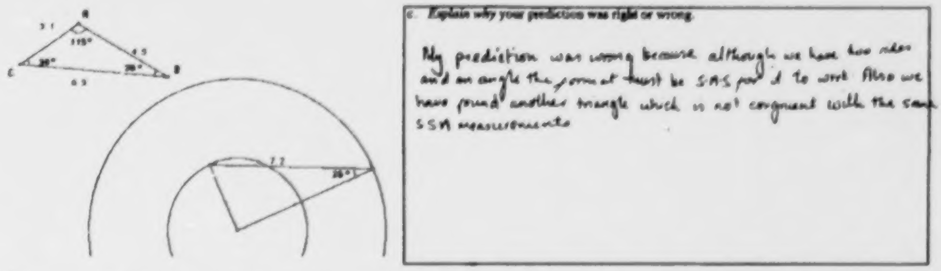

c. Explain why your prediction was right or wrong

My prediction is right because if there are fixed
sides A-C and C-B, and a fixed C-B angle, there
is only one possible length of line from A-B.
There is therefore only one angle for each
place – C-A-B, C-B-A.
It isn't possible to draw an incongruent
triangle.

Figure 5: Locating the triangle experimentally and the expanation following the activity

Although this construction strategy is clearly empirically motivated, Abby's explanation indicates that she, like Richard, was thinking generally about triangles, with, once again, no reference to specific measures. The original triangle seems to have been treated as generic and the dragging test in this case involved an exhaustive search. It is possible to see the nature of the girls' investigation reflected in the language of Abby's explanation. Whereas Richard wrote about properties being dictated, Abby suggested that they found the only "possible length", the condition is accepted in the absence of a counter-example.

Karen and Abby went on to explore Side-Side-Angle condition using a similar soft construction, by "fixing the sides" CA and CB then marking and measuring the angle corresponding to $C\hat{B}A$. This time, when the vertex on the inner circle was dragged so this angle measured 26°, it was easy to see that a triangle could be (soft) constructed in which the measure of angle corresponding $C\hat{B}A$ matched the original, but the length of the third side did not (Figure 6 presents the counter-example along with the explanation written by Karen).



c. Explain why your prediction was right or wrong.

My prediction was wrong because although we have two sides and an angle the format must be S-A-S for it to work. Also we have found another triangle which is not congruent with the same SSA measurements

Figure 6: (Soft) constructing a counter-example and the corresponding rejection of the condition

Unlike the previous pair who went on to consider other configurations of the original triangle, the investigations of Karen and Abby terminated with the location of a counter-example. They were not motivated to search for types of triangles in which the Side-Side-Angle property set would be sufficient for congruency in the same way that Richard and Tim had been. Clearly, it would have been much more tedious and time-consuming for Karen and Abby to have modified the original triangle and re-applied their investigations to check for congruency than it was for Tim and Richard, whose robust construction gave them immediate feedback. The robust construction made amenable a consideration of the domain of validity – or, more specifically, the boys were able to investigate whether the condition for congruency holds when they restricted the domain of validity from triangles to right-angled triangles. On the other hand, the existence of the counter-example was much more obvious to the pair who chose to build a soft construction.

Another difference between the two construction types was the manner in which students experienced geometrical dependency. In robust constructions, dependency is demonstrated by the fact that a relationship remains invariant through dragging. During the dragging test attention can move from general to specific as a "family" of Cabri-drawings with the same geometrical make up is produced. In soft constructions, this is not the case. Instead dragging is part of construction not verification and students observe how the dependent property becomes evident at the point in which another property is manually (and visually) satisfied. That is, the general can emerge from the specific during thorough searches for the set of loci in which the given conditions are fulfilled. In the following section, the implications of this difference – and how the way we structured students' first experiences of formal proof so that it prioritised only the first view of dependency – for the construction of valid proofs in

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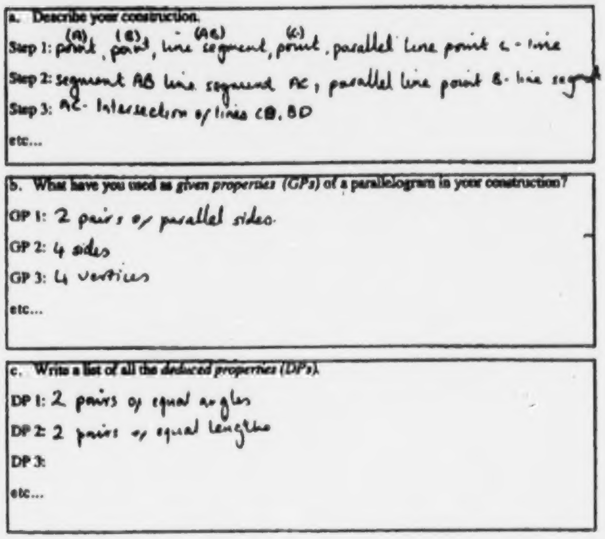
which dependent properties are deduced form those used as the givens will be considered.

Proving geometrical dependency

The writing of formal proofs was introduced explicitly in the second set of activities. The session began with a teaching episode in which a range of formal proofs (correct and incorrect) were produced and discussed. This discussion took place in the absence of any associated Cabri activity. The students were shown how formal proofs should include all the transformations (a statement coupled with a reason that justifies its truth) necessary to infer the conclusion from the givens organised into a logical chain or list. We felt that, as long as they could formulate them as a statement-reason couple, students should be free to use the few geometrical facts they already knew (this included little more that the sum of interior angles of polygons, properties of parallel lines, some basic angle and circle properties) even if strictly speaking these had not yet been proven.

The computer activities that followed this introduction revolved around the construction of special quadrilaterals – parallelograms, rectangles, rhombus and squares. The students were already familiar with these objects and had no difficulty in generating a list of properties associated with each one (much in the way described above and illustrated in Figure 1). To begin the tasks they had to define a robust construction of each of these special quadrilaterals, starting with a parallelogram, and then describing which properties they had chosen to be the givens.

In contrast to the congruent triangle activities, the approaches of Karen and Abby were very similar to those used by Tim and Richard. To avoid repetition, the girls' constructions only will be used to illustrate the interactions of both pairs. Despite their limited experience with Cabri, the (robust) constructions, and the properties (GPs) underlying them were successfully generated without any need for teacher intervention and Figure 7 presents the written descriptions produced by Karen. Having described the given properties, the pair carefully hid the construction lines used leaving only the four sides of the parallelogram visible – something that had not occurred during the congruent triangle investigation. The identification and verification of further properties (the DPs or deduced properties) was also undertaken with out great difficulties, Karen's descriptions of the two properties that she and Abby identified are also presented in Figure 7. They had marked and measured the angles of the parallelogram to verify that opposite angles were equal and used check-property confirm the congruency of opposite sides.



a. Describe your construction.

Step 1: point (A), point (B), line segment (AB), point (C), parallel line point C - line

Step 2: segment AB line segment AC, parallel line point B - line segment

Step 3: AC - Intersection of lines CB, BD

etc...

b. What have you used as *given properties (GPs)* of a parallelogram in your construction?

GP 1: 2 pairs of parallel sides.

GP 2: 4 sides

GP 3: 4 vertices

etc...

c. Write a list of all the *deduced properties (DPs)*.

DP 1: 2 pairs of equal angles

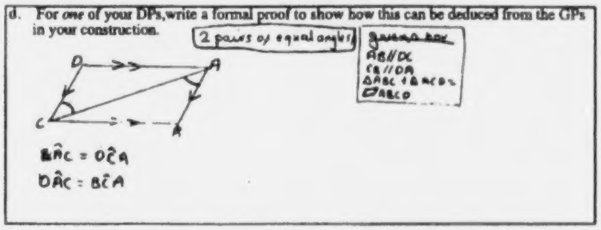
DP 2: 2 pairs of equal lengths

DP 3:

etc...

Figure 7: Separating out the given properties from those to be proven

Finally, the task required that they choose one of these verified properties and construct a formal proof to show how it could be deduced from the givens. Karen and Abby opted for the equal angles property. After a period of seemingly aimless dragging accompanied by increasing anxiety, a diagonal was finally added to the parallelogram. After more dragging, they agreed – as a result of the visual feedback – that various other angles created by the addition of the diagonal were also equal. Tentatively, and only after some prompting, Karen started to write their formal proof. Following the earlier instructions, she separated into separate boxes the given properties from the one the wanted to prove and then drew a figure of their parallelogram. Next she noted the equal angles created by the diagonal. The beginnings of this proof are shown in Figure 8.



d. For *one* of your DPs, write a formal proof to show how this can be deduced from the GPs in your construction.

2 pairs of equal angles

Givens box

AB//DC

CB//DA

ΔABC ≅ ΔACD

▱ABCD

$B\hat{A}C = D\hat{C}A$

$D\hat{A}C = B\hat{C}A$

Figure 8: The beginnings of a formal proof

Karen and Abby had enough information to come up with a proof of the chosen property but they had absolutely no idea of what to do with this information. They

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seemed to make no connection at all between the equal angles and the parallel lines they had created. This was a little surprising, our assumption had been that this was one of the few geometrical facts that the students would know about. Indeed, in the discussion that had occurred just before they began constructing the quadrilaterals this property had been considered and the girls had indicated previous knowledge of what they called "z-angles". So why were the students at such a loss?

Dynamic Geometry - good for construction but not for proof?

It might be tempting to conclude that interaction with a dynamic system like Cabri-Géomètre helps students in defining and identifying geometrical properties and the dependencies between them, but not in proving them. The fact that nearly all the students with whom we worked experienced similar difficulties in coming to formal proofs of the parallelogram properties provides further fuel for such a conclusion. But is this the only interpretation? Certainly, for this problem, their interactions during the computer activity did not facilitate Karen or Abby in making the connections necessary for the proof. Rather what seemed to happen was that after starting on the writing of the proof, the computer interactions were suspended. The way we had introduced formal proofs could be in some part responsible for this. In the teaching episode during which formal proofs had been introduced we had chosen to leave the software aside.

As we introduced the writing of formal proofs, we concentrated on the deductive only, Cabri-figures did not enter into this introduction, and perhaps the effect was that inadvertently the empirical connection had been de-emphasised – or even disallowed. In response, the students seem to have interpreted this aspect of the task as one of ritual and reproduction, just as has been observed of those following traditional approach to Euclidean proof.

A concern to reproduce teacher proof constructions may also have been the motivation behind the addition by Karen and Abby of the diagonal to their parallelogram construction. For sure, this was an action that we had encouraged. It seems that, at least in this task, the girls found it difficult to made sense of this new objects and the properties it brought about. During the construction of the parallelogram, the pair had had no difficulty in abstracting from their computer activities the mathematical properties defining their Cabri-constructions. In contrast, the addition of the extra line was not part of either the definition process or the process by which they had identified the property they were trying to prove. Might the girls have had more success if they had focussed on the relations within the whole Cabri-figure – the hidden intermediary objects as well as the visible ones (Figure 9 presents a possible scenario)? Put another way, if there had been more of a unity[8] between construction and justification procedures, would the formal proof have been easier to formulate?

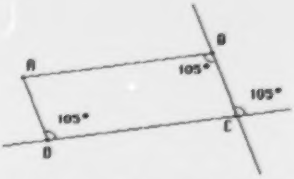

Figure 9: Manipulations of the complete construction

Whatever the motivation behind the inclusion of the diagonal, it did enable Kate and Abby to identify extra properties, potential intermediaries towards proof. The problem was that they could not make the last two steps in the proof – to explain why angles the diagonal split the angles at opposite corners of the parallelogram into two pairs of equal angles and to infer from this the equality of the opposite corner angles. Could it have been different if the triangles defined by the diagonal had figured in the construction of the parallelogram – if, for example, it had been soft constructed, with one pair of parallel lines built by means of the Cabri tool and then the second by dragging (see Figure 10). In the soft construction, the equality of all angles becomes evident at the moment BC is parallel to AD. Would this had made the reasons necessary for the formal proof more obvious?

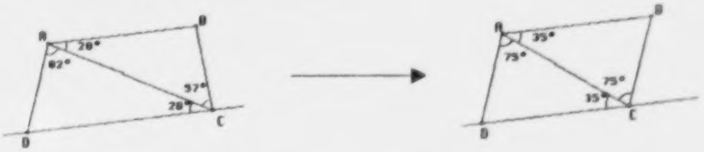

Figure 10: Soft constructing a parallelogram

Some final remarks

The issues raised in the previous section touch upon some of the complexities of designing proof activities and especially of introducing formal proof. Students' interactions with the first set of activities indicate how during the definition, manipulation and explanation of Cabri-constructions various aspects of the process of proving were encountered and students were able to negotiate these aspects using different ways of traversing between empirical and theoretical modalities. Without doubt, introduction of formal proofs in with the second activity-set was a destabilising experience for all the students who took part in the teaching experiment.

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[8] An interesting analysis of the cognitive unity of proof can be found in Mariotti, Bartolini Bussi, Boero, Ferri, and Garuti, 1997.

As the speculations about alternative approaches to the parallelogram problem imply, the path may have been a little smoother if we have structured the transition to formal proof so that the empirical-theoretical connections that characterised the construction activities were integral to this phase too and particularly if that had been a more equal emphasis on robust and soft constructions. Only further research will demonstrate if such modifications make a significant difference or whether it is inevitable that, as they attempt to enter into the particular discourse demanded for formal constructions, students will need a little time to orientate themselves. The interactions of our students with the third and final activity set in the teaching experiment, show that, even after this relatively short Cabri-experience, some were beginning to find their own ways to proof, spontaneously adopting to move between soft and robust constructions and to engage in different modes of mathematical reasoning. Further evidence of successful transitions from conjecture to proof in the Cabri-context can be found in Arzarello *et al*. (1998). There is still a need however, to better understand the various ways that this process can be negiotiated so that all of our students have the greatest chance of completing the journey from construction to proof.

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PROJECT GROUPS

Project Group PG1:
Intuitive Rules and Mathematics Learning and Teaching

Coordinators:
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Dina Tirosh, Tel Aviv University, Israel
Dirk De Bock, University of Leuven and EHSAL, Belgium
Regina Muller, Pedagogische Hochschule Erfurt, Germany

In the last decades, researchers have studied students' and teachers' conceptions and reasoning in the context of mathematics education. Many have pointed out the persistence of alternative conceptions that are not in line with accepted, scientific notions. Such conceptions cover a wide range of subject areas. Most of this research has been content specific and aimed for detailed descriptions of particular, alternative concepts. There has been several explanations of this phenomenon, one of which is the intuitive rules theory (Stavy & Tirosh, 1996; Tirosh & Stavy, 1999). The intuitive rules theory suggests that external features of the tasks, which activate certain, intuitive rules, determine students' incorrect responses to various tasks (intuitive in the sense of Fischbein, 1987). Two main strengths of this theory are: (1) It accounts for many of the observed incorrect students' responses to mathematical tasks, and (2) It has great predictive power.

The intuitive rules theory has been introduced in several research reports in previous PME meetings (Tirosh, Stavy, & Tsamir, 1996; Tsamir, Tirosh & Stavy, 1997; 1998). A growing number of PME members from different countries expressed interest in this theory. In PME23, the idea of constituting a project group on the intuitive rules, starting from PME24, was suggested by different colleagues from various countries. This group intends to explore cultural and psycho-didactical issues related to the intuitive rules theory.

During the first meeting of the project group in PME24, the intuitive rules theory will be described and discussed. The second meeting will be devoted to developing mutual related research instruments.

Product: We plan to produce a book or a special issue on the intuitive rules.

Project Group PG2:
Research on Mathematics Teacher Development

Coordinators:
Andrea Peter-Koop, University of Muenster , Germany
Vânia Santos-Wagner, Federal University of Rio de Janeiro, Brazil

This Project Group has emerged from a Discussion Group (1986-1989) which was continued as a Working Group between 1990 and 1998. One major asset of this group has been its cohesiveness and its wide representation across many countries.

In 1998 the group identified the need to investigate the various facets, implications and cultural contexts of co-operative/collaborative enterprises with respect to mathematics teacher education. While extensive literature based on the joint work of the individuals and institutions involved in mathematics teacher pre- and inservice education has been published, the choice and understanding of the terms that were used to specify the joint work (such as *working together*, *collaboration*, *networking*, *co-operation*, *partnership* or *teamwork*) as well as the quality and requirements of the co-operative and/or collaborative process itself frequently have not been the focus of attention. Consequently, the group decided to produce a book with the working title "Collaboration in teacher education – Working towards a common goal" which systematically focuses on collaborative processes in the different domains of teacher education from an international perspective. The key questions addressed in the various chapters are concerned with the ways in which collaborative endeavours can improve and enhance mathematics teaching/learning and the identification of the characteristics and conditions for successful collaboration.

The following issues are on the agenda of the two sessions during PME 24:

- A central issue will be the discussion and preparation of the concluding chapter of the book.
- Furthermore, the contributing authors have expressed the desire to reflect on the collaborative work that lead to the individual chapters and the compilation of the book. Thus, each author was invited to prepare a brief statement on the key aspects from his/her perspective in order to share reflections on and implications of the individual as well as the joint work.
- Finally, the group will have to decide about its future and a possible project. This may involve an extension of the current work by using the multimedia and database capabilities of the world wide web.

Please note: The preparation of the book and hence the work of this Project Group is in its final stage. All chapters have been submitted and are in the review process. Proposals for additional chapters therefore can not be considered.

Project Group PG3:
Understanding of Multiplicative Concepts

Coordinators: Tad Watanabe, Towson University, USA
Angela Pesci, University of Pavia, Italy
Gard Brekke, Telemarksforsking-Notodden, Norway
Anne Reynolds, University of Oklahoma, USA

Mathematics concepts within the "multiplicative conceptual field" include multiplication and division operations, fractions and decimals, ratio and proportion and linear functions (Vergnaud, 1988). Understanding of these concepts are significant milestones in children's mathematical growth. As a result, these ideas have attracted the attention of many mathematics education researchers. More recently, there seems to be a growing consensus that the understanding of these concepts requires the articulation of cognitive constructions which are closely inter-related (Harel & Confrey, 1994).

This group will provide an opportunity to mathematics education researchers who are interested in studying students' understanding of multiplicative concepts to share and discuss a variety of issues. The discussion on two themes that began at PME 23 (Haifa, Israel) will serve as the starting point of our discussion. They are: (1) proportional reasoning, and (2) teaching and learning of multiplication and division operations in various countries.

For the first theme, a group of researchers have been conducting a collaborative study, and they will share their experiences. This is motivated by the study described in Pesci (1998). For the second theme, the participants are encouraged to share the way these operations are introduced and developed in their countries, paying attention to factors such as language, conceptual principles, and the use of various representations.

One of the potential products of this Project Group is a series of cross-national collaborative studies on students' understanding of multiplicative concepts. Such studies will sure to add to the knowledge base in the field. It is hoped that the findings from these studies will result in an edited collection of papers.

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DISCUSSION GROUPS

Discussion Group DG1:
Classroom Research

Coordinators:
Simon Goodchild, College of St Mark and St John, UK
Nora Linden, University College of Bergen, Norway

The Classroom Research Group provides a forum for sharing, discussion, support and stimulation for researchers whose inquiries are based in the natural context of regular classrooms. In recent years we have focused on issues such as: software packages for collecting and analysing qualitative data; the use of video and audio recording; alternative methodologies; and use of the internet as both learning medium and research tool.

In Hiroshima we will consider how we can develop the germ of an idea into a research plan. We want to focus attention on two issues. First, we want to consider the range of research design models being used to conduct classroom research and critically review some of the newer models. Secondly, we want to take conjectures about teaching and learning in classrooms and sharpen these into well-focused research questions. We will then consider what methods - practical, realistic and purposeful - might be employed to elicit data within the context of the possible research design models.

Suppose, *for example*, the concern is to explore the interaction between students' beliefs – in self, in mathematics, and in classroom activity – and the nature of their engagement in the tasks set by the teacher. How could such a concern be formulated into tightly focused questions? What methods might be employed to elicit valid and reliable data? What analytic processes should be engaged to ensure trustworthy interpretation?
It is hoped that the discussion group will attract both experienced and novice researchers.

Discussion Group DG2:
Cultural Aspects of the Teaching and Learning of Mathematics

Coordinators: Norman Presmeg, Florida State University, USA
Vicki Zack, St. George's School, Canada

The four subgroups of this Discussion Group are as follows:

1. Theoretical perspectives on cultural aspects of learning mathematics.
2. Classroom culture and social practices.
3. Power relations (political, socioeconomic, etc.).
4. Language and culture.

As in Lahti, Stellenbosch, and Haifa, the twin aims of the meetings will be to welcome newcomers to the group as well as to continue the exploration of issues introduced in the interactions of these subgroups in previous years. This exploration has the purpose of deepening understanding of mathematics as a cultural product, and thus of how sociocultural issues impact the teaching and learning of mathematics around the world. As in previous years, the diversity of cultures represented in the group will enrich discussions; nevertheless many of the issues are common to all countries. The complexities of these issues, and possible approaches and solutions to problems, will be addressed.

Discussion Group DG3:
Encouraging Reflective Practice

Coordinators:
Anne Cockburn, University of East Anglia, UK
Francis Lopez-Real, University of Hong Kong, H.K.,S.A.R.,China
Hagar Gal, David Yellin Teachers' College, Israel

This is a new discussion group that hopes to attract delegates from all levels of mathematics education.

Its aims are:

to share methods for encouraging reflective practice which the co-ordinators have found to be effective. (These include a wide range of strategies some of which have only been possible through international collaboration.)

to consider techniques that other participants have used successfully.

to reflect on the differences between encouraging experienced and novice practitioners to become more reflective.

to summarise the above with, if it seems appropriate, a view to producing a joint paper on the topic.

Discussion Group DG4: Exploring Dilemmas of Research on the Social Aspects of Mathematics Education

Coordinators:
Elsa Fernandes, University of Madeira, Portugal
João Filipe Matos, University of Lisbon and CIEFCUL, Portugal
Madalena Santos, CIEFCUL, Portugal

This discussion group continues the interest of the "Social aspects of mathematics education" group in PME 22 and 23 about research methodology. Despite of the previous work, there is still a need to deeply explore the endeavor of producing knowledge about these social aspects. Furthermore, it is imperative for the international research community in mathematics education, represented in PME, to tackle in serious ways the social nature of the object of study we deal with (Lerman, 1998).

For this discussion group we propose to explore the emergence of *dilemmas* in mathematics education research from a social perspective. These dilemmas are critical situations faced in the process of a study, where the researcher has to make decisions in which the basic assumptions of mathematics education investigations are strongly questioned. Based on the experience of the group coordinators in two research projects adopting cultural and sociopolitical approaches, we will discuss four main dilemmas. The *dilemma of the mathematical specificity* refers to the tension between the high or low priority and importance that a mathematical point of view is given, versus the high or low priority and relevance given to the social, cultural and political settings and relations in which the learning and teaching of mathematics are embedded. The *dilemma of the scope* addresses the issue of the level and unity of analysis of the investigation and the narrowness or openness of its focus. The *dilemma of the scientific distance* refers to the complexity of the relationship between observation and participation and highlights the issues of objectification of the people "researched". Finally, the *dilemma of the relevance of mathematics education* questions the very basic assumption of all research in the discipline about the importance of mathematics teaching and learning in society, when these practices are seen from the point of view of students –and not the researcher.

The work of the group will be structured around the exploration of the meaning of these dilemmas and its possible causes, as well as the identification of other dilemmas that participants had experienced in their endeavor. The final aim of the group will be to build a landscape of the potential critical situations that emerge in research on the social aspects of mathematics education, and of the possible ways of dealing with them.

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Discussion Group DG5:
Imagery and Affect in Mathematical Learning

Coordinators:
Lyn D. English, Queensland University of Technology, Australia
Gerald A. Goldin, Rutgers University, USA

Representational systems in mathematical learning include not only external structured physical situations, but also internal systems that encode, interpret, and operate on mathematical ideas (Goldin & Janvier, 1998). We focus on imagery, affect, and their interplay: with internal systems like natural language, formal notations, and heuristic planning and control, and especially with each other. The traditional view of mathematics as an abstract, formal discipline has tended to relegate visualization, metaphor and metonymy, emotions, and the relation between feeling and mathematical imagination to the status of incidental concomitants. Yet we have a case for the centrality of imagistic reasoning: analogies, metaphors, and images in mathematical learning (English, 1997; Presmeg, 1998); Lakoff and Nunez (1997) even aim to recast the foundations of mathematics in terms of metaphorical image schemas. The essential role of affect has also been stressed (McLeod, 1992); affect may be taken as representational, encoding information and influencing learning and performance, and it may be the most fundamental and essential system in powerful mathematical learning and problem solving (DeBellis & Goldin, 1997).

The purpose of this discussion group is to explore the nature and role of affective and imagistic representational systems in mathematical learning and problem solving. The first session will begin with brief presentations: by Lyn English on analogies, metaphors, and images in mathematical reasoning, and by Gerald Goldin on affect and meta-affect in problem solving. Participants are encouraged to cite examples of imagery, affect, and their interplay in children and adults doing mathematics, for discussion and interpretation by the group. Some of the difficult issues in the empirical investigation of these topics through classroom observations and structured clinical interviews will be discussed, and key research issues identified that could form the basis for a future PME project group.

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Discussion Group DG6:
Stochastical Thinking, Learning, and Teaching

Coordinators: John Truran, University of Adelaide, South Australia
Jenny Way, University of Cambridge, UK
James Nicholson, Queen's University at Belfast, UK
Mario Barra, Universita La Sapienza Roma, Italy

The discussion group on Stochastical Thinking, Learning, and Teaching will focus this year on the issue: *The Relationship between Stochastical and Mathematical Thinking, Learning, and Teaching*. It is our intention to approach this theme from multiple perspectives, including:

1. Philosophical, in terms of the perceived boundaries of the disciplines.
2. Historical, in terms of the developments of the disciplines.
3. Educational, in terms of the positioning and implementation of the teaching and learning of stochastics within school and tertiary curricula, including such fundamental issues as teacher development, assessment, and technology.
4. Psychological, in terms of the specific cognitive and sociocultural processes involved in the teaching and learning of stochastics.
5. Research, in terms of cross-fertilization of theoretical frameworks and methodologies.

The following short contributions will be presented in the meetings to provide a focus for discussion, and to serve as an opportunity for feedback from other participants.

1. Perspectives from students and teachers on the differences in thinking in Mathematics and Statistics.

James Nicholson, School of Psychology, Queen's University of Belfast, N. Ireland.

2. Relations between probability and others languages of science, particularly referred to the pitagorean "aritmo-geometry" carried on d-dimensions and in the continuum and discrete spaces, closely connected: a didactic use.

Mario Barra, Dipartimento di Matematica, Università "La Sapienza", Roma, Italia.

3. The Relation between Pattern and Randomness.

Jenni Way, NRICH Online Maths Project, University of Cambridge, UK.

The meetings will be closed with a brief discussion on identifying and pursuing specific items for potential collaboration in the area of stochastics teaching and learning, further research and its dissemination.

A mechanism for electronic communication between potential participants exists through the PME Stochastics Teaching and Learning Newsletter, which has been circulating for four years, and through the STL discussion group's website (http://www.beeri.org.il/stochastics).
To join, write to: dani.ben-zvi@ weizmann.ac.il.

Discussion Group DG7: The Importance of Matching Research Questions and Methodology to the Reality of Researchers' Lives

Coordinator: Kath Hart, University of Nottingham, UK

Researchers, have for many years, chosen the methodology to provide evidence for a research problem in order to suit the problem and their own bias. Students from countries which have no mathematics education research community enrol in foreign universities to study for a higher degree and if successful return home to be the leaders in the field. Their reputation as a researcher, their expertise as a future supervisor and the knowledge they can pass on to their students are to a large extent dependent on the doctoral /masters research and the thesis. Should the methodology they use be chosen to reflect the context in which their career is likely to be led?

We hope that government policy in mathematics for schools and colleges is informed by the research carried out by mathematics educators. There is a tendency in some countries to regard education research carried out in universities as esoteric and of insufficient practical use to guide policy making. Is the methodology used to blame? What criteria do we adopt when choosing how we will do research?

The group will start by discussing the methodology used in various pieces of research reported in short articles, then move onto statements made by reviewers and examiners which display a belief in what is acceptable research practice. Finally, we will consider at least two government or state policies on research. The aim is to provide a forum for discussion of research methodology in the context of 'who do we want to tell'?.

Discussion Group DG8:
Theory of Embodied Mathematics

Coordinators:
Laurie D. Edwards, St. Mary's College of California, USA
Rafael E. Núñez, University of Fribourg, Switzerland

Recent theoretical and empirical work in cognitive science has generated results with implications for our understanding of mathematics and mathematical thought. One line of research has investigated the ways in which thought is a fundamentally embodied phenomenon. This work includes research into basic innate capabilities of humans (and some other animals), including very simple abilities to distinguish small quantities as well as perform a limited kind of arithmetic. Another line of inquiry has focused on fundamental conceptual mechanisms that underlie human understanding, including unconscious mappings between conceptual domains that permit the transfer of inferential structures.

These two lines of inquiry come together when considering how it is that human beings are able to think mathematically, and when looking at the subject matter of mathematics from the point of view of cognitive science. The purpose of this discussion group is to consider a number of questions that arise when we accept several propositions: that mathematics, as we know it or can know it, exists by virtue of the embodied mind; that all mathematical content resides in embodied mathematical ideas; and that many of the most basic, as well as the most sophisticated, mathematical ideas are metaphorical in nature.

A certain portion of the discussion group will be devoted to explicating these ideas, but the bulk of the time will be spent in discussing their implications. In particular, we will look at such questions as:

- how are mathematical ideas grounded in our experience?
- how do the mechanisms of conceptual metaphor and conceptual blends work to permit the construction of specific mathematics content (e.g., number, infinity, continuity, complex numbers)?
- what accounts for the apparent universality, stability, consistency, and generalizabiity of mathematics?
- how do social and cultural factors interact with basic conceptual mechanisms in the historical development and learning of mathematics?

The discussion group will be organized around readings drawn from a new book by Lakoff & Núñez on embodied mathematics. The readings will be made available in advance, on the Internet, as well as during the first session, during which time the basic framework will be introduced. During the remaining time, specific topics will be discussed in small group format, with a concluding general discussion.

Discussion Group DG9:
Underrepresented Countries in PME:
The State of Mathematics Education

Coordinators:
Bernadette Denys, I.R.E.M. - Université Paris 7, France
Paola Valero, Royal Danish School of Educational Studies, Denmark

In PME NEWS-May 1999, our subgroup in PME's international Committee, devoted to pro-actively strengthening and promoting the activities of PME in underrepresented countries, expressed its concern about the policies of our community in this area.

In previous PME Conferences, the underrepresented Discussion group meeting explored the cultural, social, economic and political climate that supports the development of mathematics education, as both research and practice communities in under- or non-represented countries in PME.

The situation has not evolved much in recent years, as it relates to the knowledge that the international community of research in mathematics education, represented in PME, has gathered about the state and conditions of the discipline and practices in underrepresented countries. Therefore, our group proposes to develop a plan for collecting experiences from under- or non-represented countries. Information about the state of mathematics education in those countries will be diffused in the ways that the participants in the Discussion Group participants will choose.

The objectives are to bring to light the real situation of mathematics education in under- or non-represented countries and to point out the relevant foci for research in mathematics education in those national communities and contexts.

SHORT ORAL COMMUNICATIONS

RELATION BETWEEN DISPOSITION OF TASK AND STRATEGY OF COMPUTATIONAL ESTIMATION ADOPTED BY STUDENTS 10 TO 17 YEARS OLD

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The purpose of this research is to make clear how students' selection of computational estimation strategy is influenced by the difference of disposition of calculation tasks. In this study, three kinds of calculation tasks were given to 220 students from 5th graders (10 year olds) to 2nd grade students in high school (17 year olds). Concerning the computational estimation strategy which students used, the next points were expected. For solving Task (A) 38×99, it is easy to use the rounding strategy such as 40×100 or 38×100. For Task (B) 9250÷25, strategy of 9000÷30 or 10000÷25 is effective, and for Task (C) 0.24×439, it is available to use either the strategy of dividing both numbers by 5 or the strategy of substituting 0.24 with 0.25 or 1/4.

The estimation strategy adopted by students was classified in the following 6 kinds of categories. (1) Exact calculation. (2) Exact calculation at first, then rounding the answer. (3) Adopt a simplified calculation method, and get an exact answer, e.g. in case of Task (B), 9250÷5 → 1850÷5, or 9250÷100×4, 9000÷25 = 360 →250÷25 = 10 →360 + 10. (4) Round only one of the two figures, then calculate. (5) Round both figures then calculate. (6) Adopt a simplified calculation method, and do estimation, e.g. in case of Task (B), 10000÷25×9, or in Task (C), 440×25÷100 = 440÷4 = 110, and 25÷100×450 = 25÷2×9 = 25÷2×10 = 125.

The following results were clarified.

1. Strategy commonly used by students for all tasks were "Round both figures then calculate" and "Exact calculation".

2. Strategy difference shown by students was influenced by the disposition of tasks. In Task (A) 38×99, strategy of "Round one of the two figures" and "Round both figures" were adopted at the high rate (61.8% in total). For Task (B) 9250÷25, strategy of "Adopt a simplified calculation method, and get an exact answer" was mainly used compared with other tasks (10.9%). In Task (C) 0.24×439, many students intended to make a rough estimate by "Adopt a simplified calculation method, and do estimation" strategy (11.4%).

3. Students' age affected their strategy use. Elementary and high school students used "Exact calculation at first, then rounding the answer" more. Some high school students stuck to the exact calculation, however, some prefered to use "Adopt the simplified calculation method" strategy.

4. According to the analysis of calculation errors, two kinds of error were found; calculation error and place value error. Task (A) and (B) included "calculation error" but Task (C) included both kinds of error.

It was showed that estimation strategies adopted by students were influenced by the disposition of calculation tasks and calculation errors were introduced by their estimation strategies.

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A CROSS-CULTURAL SELF-SELECTION PROCESS OF ELEMENTARY PRE-SERVICE TEACHERS

Miriam Amit, Ministry of Education and Ben-Gurion University, Israel

A questionnaire based on Causal Attribution Theory (Fennema et. al., 1977), aimed at examining confidence in personal mathematical ability, was administered to pre-service elementary teachers who belong to three different cultures:
(1) Native born Israelis, mostly secular, who were raised in Israel; (2) Arabs, specially those who were raised in a Beduine culture; (3) New immigrants from former Soviet Union.

Based on our previous acquaintance with the different types of population (Amit, 1988; Amit et. al, 1989), we started with two conjectures: (a) there will be no gender differences between Beduin boys and girls in attribution of success or failure; (b) female new immigrants from the former Soviet Union will show a higher level of self confidence in mathematical ability than their Native-Israeli and Beduin colleagues.

The first conjecture was based on the assumption that girls who "rebel" by rejecting the traditional position of the Beduin woman and opt for further studies and a teaching career must possess a very high degree of self-esteem and confidence in their ability. The second conjecture rested on the very high proportion of women with degrees in engineering and other technical professions among new immigrants as compared with Israelis; we reasoned that this situation reflected a general perception of equal mathematical abilities among men and women in the former Soviet Union. These two conjectures were almost demolished.

Results show that: (1) Although Bedouin girls "rebel" by opting further studies, there were still slight differences favoring Bedouin boys regarding self-esteem and confidence in mathematical ability; (2) There was no significant difference in self-confidence in mathematical ability between pre-service teachers among women immigrants from the former Soviet Union and those in other two aforementioned population. These findings, that emerge also from personal interviews with some of the participants in the study, can be explained either by a "self-selection" process or by a socialization process, as we shall portray in the session.

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Fennema, E. & Sherman, J. (1977), Sex-related differences in mathematics achievement, spatial visualization and affective factors", *American Educational Research*, 4, pp. 51-71.

THE INFLUENCE OF THE SEQUENCE OF INFORMATION IN THE SOLUTION OF A WORD PROBLEM

A.Archetti①, S. Armiento②, E. Basile③, L. Cannizzaro④, P. Crocini⑤, L. Saltarelli⑥

①S M Buonarroti, Roma; ②S M Amaldi, Roma; ③S M Platone, Roma; ④Dipartimento di Matematica, Università Roma 1, Roma; ⑤1° Circolo, Aprilia; ⑥3° Circolo, Aprilia.

Reasearch question was if and how the solving of a word problem is influenced by the coincidence of three temporal structures: the sequence of the informations in the text, the sequence of the real actions evoked and the sequence of steps for processing data in solving strategy. Our research hypothesis was that the reconstruction of the action takes place in an implicit manner so if the temporal sequence of information is different from that of the real action, the problem represents an extra difficulty to be overcome. And data ratified with evidence the hypothesis (A complete report is to appear in January issue of L'Insegnamento della Matematica e delle Scienze Integrate).

Research design: five problems has been submitted to 171 children, aged 11 to 12 in the last year of elementary school and in the first year of middle school, divided into four problem solving ability levels and into three *formal* homogeneous groups. Problems had been possed in three linguistic formats to coincide with different time structures: in the TTC (Chronological Text Time) version the text time coincides with the chronological time; in the TTS (Text Solution Time) version the text time coincides with the sequence which was to be used in the solution algorithm;in the TTD (Different Text Time) version the time of the text is different from the previous. Other linguistic characteristics of the wording of the problems has been left unaltered (the question, in explicit form, always placed at the end).

Observations. 1.The temporal dimension doesn't have the same role in all problems. 2.The change in the text modifies also the reciprocal position of information. 3.There are several TTD versions that should be investigated. And even the TTC or TTS versions are not always unique.

At present we are examining protocalls to find traces of the genesis of the mental processes of the able problem solvers with respect to the temporal reconstruction of the action. The question is if such reconstructions are done implicitly or are skipped altogether and if it may be described in term of *prospective structure as identified by* Nunokawa (1993).

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Students' confusion between verbal and visual representations of geometric figures in Dynamic Geometry Environment[*]

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In this paper, we discuss some discrepancies between two ways that students appear to represent a quadrilateral (and presumably other geometric objects) to themselves—one has a more holistic, impressionistic character, and resembles a *visual* image, and one has more the character of a *verbal* or propositional description. We consider two possible ways that high school students might develop their internal representations of plane geometric objects and we explore how they reconcile these representations with conflicting external cues they get in a Dynamic Geometry Environment (DGE). In our conclusion, we propose strategies for helping students to consolidate their impressionistic images of concepts with their propositional ones (definitions), and to improve their ability to formalize information that they gain implicitly in visual (or other) images.

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[*] This paper was supported by NSF grant RED-9453864, EDC's Epistemology of Dynamic Geometry project. Opinions and conclusions expressed here are not necessarily shared by the National Science Foundation or its staff.

PLUS, AND AND ADD: ADDITION AND ENGLISH ADDITIONAL LANGUAGE LEARNERS OF MATHEMATICS

Richard Barwell, University of Bristol, UK.

Research into English Additional Language (EAL)[1] learners of mathematics has generally focused on language and attainment: there seems to have been little investigation of the *process* of learning and understanding mathematics. This paper outlines two small exploratory studies. Investigating the learning of EAL students in mathematics involves several dimensions of complexity, including the learner's intellectual and linguistic development. Vygotsky's (1962) theory of learning based on the development of word meanings offers a framework which addresses these factors.

First exploratory study.

L (EAL) and V (monolingual) worked together on an arithmetic activity. A count of the words used during the task to refer to addition reveals that in this activity L seems to rely on the word 'plus' (see Table 1). Why does L use 'plus' as his word for *addition* rather than an alternative? The other students in the study generally used 'add'. Perhaps L, who is new to the school, has carried 'plus' with him from elsewhere. How would this affect L's communication in the mathematics classroom? What contexts or previous experiences do EAL learners associate with classroom language? To explore these questions, the second study focused more explicitly on word meanings.

Word used	plus	add	and	times	**Total**
L	9	1	0	3	**13**
V	5	7	4	0	**16**

Table 1: addition words used by L and V.

Second exploratory study.

As Vygotsky's (1962) theory was found to be limited by the lack of any framework for analysing word meanings, Saussure's (1974) concepts of paradigms and syntagms were introduced, which Luria (1981) has linked to the hierarchical relationship between concepts, and thus to Vygotsky's (1962) 'scientific concepts'. In the second study two EAL students were asked to identify addition words (on cards) which 'go together'. While discussing LESS THAN and MORE THAN, one student, G, provided a textbook example of meaning derived from paradigmatic opposition: *yes/ cos like the opposite/ MORE THAN LESS THAN// so like MORE THAN means give me a number that is MORE THAN/ nine/ and LESS THAN means/ give me a number that is LESS THAN nine*

G also makes sense syntagmatically: 'give me a number that is ... nine', surely an example of 'teacher-talk', suggesting that for G, the meanings of MORE THAN and LESS THAN are closely related to the discourse of the mathematics lessons in which they are used, raising questions about how this may affect G's mathematical learning.

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1. English additional language (EAL) refers to any learner in an English medium learning environment for whom English is not the first language and for whom English is not developed to native speaker level

CHARACTERIZING GENERATIVE AND SELF-SUSTAINING TEACHER CHANGE IN A CLASSROOM PRACTICE THAT PROMOTES STUDENTS' ALGEBRAIC THINKING

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Traditionally, the mathematical experience (and hence classroom practice) of most elementary teachers has been deeply oriented to arithmetic and computation, not the activities of generalizing and formalizing that are now widely believed to be an essential part of elementary mathematics (Romberg & Kaput, 1998). The larger goal of our work, part of which is reported here, is to develop elementary teachers' abilities to identify and strategically build upon students' attempts to generalize and progressively formalize their thinking via more powerful mathematical symbol systems. To this end, one of our immediate goals was to consider the classroom practice of a teacher whose instructional purpose was to promote students' algebraic thinking and, in particular, to understand how that teacher's practice reflected generativity and sustainability in the development of students' algebraic thinking. We report here results of our current work with a 3rd-grade classroom teacher (Jan – pseudonym) who has participated with us in a two year, district-wide project designed to integrate the various forms of algebraic thinking (Kaput, 1998) into elementary classrooms. In particular, we have spent the past academic year in her classroom, observing her two-period mathematics class approximately 3 days per week, in order to characterize generative and self-sustaining teacher change in the context of a classroom practice which extends beyond arithmetic to algebraic activity. Our data consists of classroom field notes, audio recordings and Jan's reflections.

From our analysis, we see the following characteristics emerging as indicators of generative and self-sustaining teacher change: (a) an ability to generalize an activity (what we describe as "activity engineering"); (b) the seamless and spontaneous integration of algebraic conversations in the classroom; (c) the spiraling of algebraic themes over significant periods of time; and (d) the integration of independently valid algebraic processes in a single mathematical task. We will present classroom vignettes that concretely illustrate these characteristics and discuss the circumstances that led to this kind of teacher change.

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ARE YOUNG CHILDREN ABLE TO REPRESENT NEGATIVE NUMBERS?

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Oxford Brookes University

This study aimed to investigate how distinct systems of signs, such as spoken language and use of manipulative material, influence young children's ability to represent negative numbers.

Nunes (1993), in a series of studies that explored the role of representation in the understanding of negative numbers, observed that the written representation was the cause of most misunderstandings that adults and children had in operating with negative measures.

In the present study 60 seven and eight-year-old children were asked to solve 12 problems that involved negative numbers in the context of games. The children were randomly assigned to four experimental groups. Two of the groups were requested to solve the problems orally and the other two solved the problems choosing some sort of manipulative material (marbles, sticks, coloured papers, rulers, paper and pencil or coloured pens). Half the children solved problems that resulted in negative measures and the other half solved problems that resulted in negative relations.

In all groups children were able to develop adequate representations for negative numbers but the experimental groups that solved the problems orally performed significantly better ($p<.05$) than those that used manipulative material. There was also a significant interaction ($p<.05$) between the form of representation (oral or use of material) and number meaning (measure or relation). Amongst the children that solved the problems orally those that dealt with negative measures performed better than did those that dealt with negative relations. The children that used material performed similarly whilst dealing with both number meanings for negative numbers.

The ability to externalise their representations orally and to manipulate these representations with much more ease than when requested to use other modes of externalisation, suggest that in the classroom close attention be paid to the role of systems of signs in the understanding of concepts. Pupils are able to develop their own systems of signs and must be encouraged to employ these signs in problem solving and to discuss how they may be used.

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INCLUSIVE SCHOOL AS A REALITY: THE ROLE OF PEER INTERACTIONS IN MATHS CLASSES FOR PUPILS WITH EDUCATIONAL SPECIAL NEEDS

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The importance of social interactions has been studied during the last three decades. Under the influence of Vygotsky theory (1962, 1978), the relevance of the context and social interactions in pupils' academic performances became more evident. Working in a collaborative way promoted pupils' performances in the Maths classes both for symmetric as well as assymmetric dyads (César 1997, 1998, 2000; César e Torres, 1997). Working in the zone of proximal development was a very effective way of respecting and dealing with diversity. In the last decade Schubauer-Leoni (1986) stressed the importance of the didactic contract in the process of knowledge appropriation. Changing the traditional contract is an essential step in order to implement innovative practices in the Maths classes.

In the last six years we developed an action-research project (5th to 12th grades) whose main goals are to give opportunities to all children to develop positive attitudes towards Mathematics, a positive self-esteem, to promote their socio-cognitive development and their school achievement. This project is called *Interaction and Knowledge*. The data that we are presenting now were collected through participant observation (different observers), audio and videotaping, questionnaires (to all pupils) and interviews (to selected pupils).

The most relevant result was that for many pupils this was the first opportunity to experience a success related to Maths. The analysis of an interaction illustrates the role of these innovative practices in the inclusion of pupils with educational special needs (handicapped; learning disabilities). Some phrases from pupils' questionnaires and interviews also stressed the importance of this kind of work in order to accept their peers' diversity and to promote their mathematical performance. In some schools which had a lot of pupils with special needs diversity played a more important role and peer interaction was essential in order to guarantee the integration and acceptation of all the elements of each class.

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A THINKING MODEL OF MATHEMATICS CONJECTURING

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This study aimed to investigate students' thinking processes of mathematics conjecturing. Five grade eleven students, 5 undergraduate students and two mathematicians were interviewed with a special survey in the study. The survey included two parts: The Mathematics Conjecturing Test for students and The Expert's Interview Questionnaire for mathematicians. It was found that the pictorial representations of students' thinking processes of mathematics conjecturing could be constructed as a unified model. It was further found that students' conjecturing model and mathematicians' conjecturing model are compatible. The conjecturing model contained two directed cycles, an inner cycle and an outer cycle. The inner cycle represents the process of refining the primitive conjecture and the outer cycle represents the process of rejecting the primitive conjecture and reforming a new conjecture. The conjecturing process appears to move dynamically and recursively between four stages: guessing, checking, confirming and refuting. When students work on tasks of guessing an unknown conclusion, the model reveals that the starting point of students' thinking and their thinking paths are more complicated than the corresponding representation on tasks of judging the correctness of a proposition. This model that we constructed can be used to study the difficulty of the tasks and to evaluate the quality of individual thinking.

ASSESSING HOW CHILDREN LEARN HIGHER ORDER MATHEMATICAL THINKING

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The Study

The project aims to study the development and assessment of higher order mathematical thinking for primary four (age 10) children in Hong Kong. Romberg, Zarinnia, & Collis (1990) commented: All learning involves thinking, but in the past, most instruction focused on learning to name concepts and follow specific procedures. Now the emphasis for all students must shift to communication and reasoning skill. Although these skills resist precise definitions, they are now popularly called "higher order" thinking skills.

Methodology and Result

The authors go to two schools and teach higher order thinking which match with the objectives of the mathematics curriculum. The methodology of the study is as follow:
A class of primary 4 student from two schools were selected and examination results of the students are recorded. A pretest using test items related to TIMSS is used and the author taught the classes for 8 weeks with learning tasks, which meant to catalyze higher order thinking. Tests were conducted every three weeks to see the development of the thinking skills of the children. Finally, a test was given to all classes, to see if there is any significance in changes in the mathematical achievement both in the higher order thinking and the subject achievement.

Logically, the results of the performance of these two classes should out perform the other classes after the 8 weeks instruction. The analysis showed that not only the two classes were consistently better performed than other classes, but also that the performance measured in SD range further away from zero, meaning that the performances of the two classes improved at a greater pace than the others.

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MISCONCEPTIONS IN 3-D GEOMETRY BASIC CONCEPTS

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Basic concepts like "straight line" and "plane" and concepts of interrelation between them like "perpendicular" and "parallel" are usually known in the context of plane geometry. The extension of these concepts to three-dimensional space does not change their basic meaning but it enlarges the variety of possible relationships between them.

These new possibilities need an ability of visualization, which is often quite limited in students who are used to seeing everything in a plane. Even if the students are aware of the existence of different plans and directions, they tend to see only one plane at a time. Moreover, some statements that were true and well known in plane geometry are no longer true in 3-D geometry. This kind of conflicts and confusion is usually a ground on which misconceptions could easily arise.

This research is an attempt to locate and to analyze these misconceptions among prospective teachers in a college of education.

The study consists of a questionnaire, which was given to 272 mathematics majoring students and was followed by group discussions. During the discussion, the students used models and were encouraged to argue, to explain and to illustrate their opinions. Some of those discussions were video or audio taped. In addition, some students were orally interviewed.

The research has also pedagogical aspects, such as:

- The awareness of the students to their own misconceptions and to the possible reasons that caused them.
- The role of the models and the role of the discussion in trying to overcome those misconceptions, etc.

In the oral presentation some of the findings will be presented and analysed, with examples of episodes that took place during the discussions.

Primary Teachers Interpretations of Graphical Representations Teachers: Differences between Static and Dynamic Data Analysis Tools

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For the past three years, we have been investigating how primary teachers' interpretations of statistical graphs and diagrams evolve as they work through computer-based data-handling activities using Tabletop. Initially, our activities were structured into sets, each organised around the use of a particular data analysis tool. So, for example, tools to construct frequency graphs were introduced in the context of questions about the distribution of single variables and, in activities involving relationships between variables, teachers were directed to construct scatterplots. We found that teachers quickly learnt how to construct the different graphs and to change the variable under investigation. Reading the graphs was somewhat more problematic, with, not surprisingly, teaching finding scatterplots considerably more difficult to interpret than frequency graphs.

Through our observations of teacher interactions with the software, we became aware that we had not taken into account the differences between static paper-and-pencil graphs and the dynamic representations possible in Tabletop. Essentially, the Tabletop tools had been presented as economic and efficient means of constructing conventional graphs and figures and we had not considered how the tools of the software might enable new mediations upon statistical ideas. This suggested a new research focus, that of identifying if and how the dynamic Tabletop representations: aid teachers in making connections between different types of graphs and figures; provoke them to extend their intuitive strategies to include the *institutional meaning* of particular statistical concepts (Batanero *et al.* 1998); and, in the long-run, support them in adopting statistical perspectives in which they both co-ordinate and differentiate between group properties and individual cases (Konold *et al.*, 1996).

In this presentation, we intend to describe how our teachers chose data analysis strategies which rarely corresponded to "standard" solutions and often would have been impossible in a paper-and-pencil context. For example, to consider associations between two variables, they tended to construct frequency graphs for one variable and then visually modify the data points according to a second variable or label the points with its value. A major advantage of working with the Tabletop software for primary teachers seems to be that they could construct global visions of group properties whilst preserving access to individual data.

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THE IMPORTANCE OF PRACTICAL FLUENCY IN PROBLEM SOLVING

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In my report I am going to present some quantitative and qualitative data from my research. This material will illustrate the importance of practical fluency in the ability to apply knowledge flexibly when solving mathematical problems. Following a teaching program based Gal'perin's principles of activity theory [see Haenen (1996) for a summary of Gal'perin's work], a series of tests and a dynamic assessment were carried out according to a method outlined by Ferrara (1987). I thus looked both at what the children could do unaided and at what they could do in co-operation with a more capable adult. I found that the greater the amount of help that was needed in completing the practice papers, the lower the gains that were made during practice. Also, the amount of help needed was an important factor in predicting these gains and depended on the degree of fluency of mental actions which had been developed.

I will present a video record from one of the lessons to show how a simple task became problematic when fluency was not developed. The problems of a pupil who scored poorly in the tests at the end of the program, and whose dynamic assessment indicated a low level of fluency, will be summarised in terms of: ORIENTATION (knowing what to do next); EXECUTION (an ability to carry out the action); and CONTROL (checking the action against a model). The developing buds of her ability (to use a gardening metaphor) were, however, clear in the video record and she was able to make great progress in catching up given dynamic assistance in practice papers during the dynamic assessment. The guidance I gave substituted for the fluency she lacked. In this way her achievements were not only measured sympathetically, they also revealed ways to help her improve. I will argue that simply looking at test data in isolation could not provide this diagnostic help.

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SPACE AS A LOCUS OF POLY- AND TRANS- DISCIPLINARITY IN MATHEMATICS, ART AND GEOGRAPHY

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1. A multidisciplinary group of educators and researchers in education engaged in the study of the ways that students in compulsory education structure space, by identifying and comparing the opportunities presented by each of the disciplines that contribute to the development of spatial competency in students.

The group established two concurrent directions for its work:

- the first, epistemological in nature, involved the expression of relationships with space, both within the specific compass of each of the three disciplines, and in ordinary communication.
- the second was concerned with didactic and pedagogical analyses of the implementation of the circumstances of spatial learning in contexts of communication.

2. The project was multidisciplinary, interdisciplinary and transdisciplinary. These three dimensions were supposed to become apparent spontaneously in the course of our work. We made several assumptions:

- the first assumption was that the disciplines, goals, and language of the three fields were compatible. However, as our research and the analyses progressed, we became increasingly aware of the incommensurability of our fields of discipline. Indeed, we have practiced our respective disciplines (at an elementary level); we have revealed the nature of spatial perception in each of them, and we have experienced the obstacles presented by the terminology and concepts peculiar to them.
- the second assumption was that it was "necessary" for students to arrive at a "unifying recombination" of their conception of space that would proceed from the shattering of the individual disciplines.

3. The obstacles we encountered led us to view our assumptions as myths. However, the various cognitive, perceptional, and linguistic components of spatial cognition prompt us to transcend the boundaries of disciplines. Like Edgar Morin, we are aware of the complexity of the organization of knowledge: "Developing the ability to contextualize and globalize knowledge is becoming an educational imperative."

WHAT MIGHT BE LEARNED FROM RESEARCHING VALUES IN MATHEMATICS EDUCATION?

Gail E. FitzSimons, Wee Tiong Seah & Alan J. Bishop, Monash University

Philip C. Clarkson, Australian Catholic University

In 1999 the Australian Research Council began funding a three year project which had the goals of: (a) investigating and documenting mathematics teachers' understanding of their own intended and implemented values, (b) investigating the extent to which mathematics teachers can gain control over their own values teaching, and (c) increasing the possibilities for more effective mathematics teaching through values education of teachers, and of teachers in training. Details of this qualitative, interpretative research project are available on the Values and Mathematics Project [VAMP] website: http://www.education.monash.edu.au/projects/vamp/

Values in mathematics education are the deep affective qualities which education aims to foster through the school subject of mathematics and are a crucial component of the classroom affective environment. While accepting that values, beliefs, and attitudes are dialectically related, our concern is with the values of mathematics, mathematics education, and education in general, rather than more global values such as social, ecological, moral and so forth - although these are by no means incompatible. (See Bishop, FitzSimons, Seah, & Clarkson, 1999, for more theoretical perspectives.)

Problems and tensions encountered during the first year of the project were mirrored in a general reluctance by teachers to be involved, and included: (a) teachers' sensitivity to being judged personally and professionally, (b) the need to define operationally what we meant by values, (c) the relevance of the project to the everyday work of teachers. In addition we were sensitive to factors impinging upon the conditions of teachers' work, such as: (a) the ideological purposes of schooling, (b) the sociocultural frameworks of the state and the particular school, and (c) the community of practice within any particular classroom.

The means adopted to overcome these challenges were centred around the need to inform teachers and to gain their confidence, as was the strategy in a parallel project in Taiwan (F-L Lin, personal communication, January, 2000). To this end we conducted a series of professional development workshops incorporating as focal points for discussion the decision-making processes of classroom teachers through: (a) video clips, (b) vignettes, and (c) comparative sample textbook analyses.

From the above we learned that teachers are committed to: (a) upholding their own and their students' integrity, (b) the personal, social and educational development of their students, and (c) the prime importance of context in its fullest sense.

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THE METALEVEL OF COGNITION-EMOTION INTERACTION

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This paper examines the self-referential aspects of human cognition and emotion and proposes a new conceptual framework for it. This domain will be divided into four different categories: 1) cognitions about cognitions (metacognition), 2) cognitions about emotions (emotional cognition), 3) emotions about cognitions (cognitive emotions), and 4) emotions about emotions (metaemotions).

1) Metacognition is an established concept and it will not be elaborated here.

2) We are not always aware of our emotions, as we are not always aware of the shoes in our feet. However, we can become aware of our emotions, reflect, and sometimes even control them. Furthermore, we often know how we would feel in certain situation. All this awareness, reflection, control, and knowledge of our emotions are cognitive processes - emotional cognition.

3) Emotions exist in relationship with goals. Cognitive emotions are emotions that are related to cognitive goals. Cognitive goals may be explicit, like when one wants to remember a fact or a. Sometimes the goal may be vague, like 'to understand' a topic. Cognitive goals direct human cognition through such cognitive emotions as surprise, curiosity, frustration, and pleasure.

4) Metaemotions are emotions that are related to emotional goals. Presumably all humans share the goal to experience pleasure and avoid unpleasant emotions. There are, however, different norms and individual coping strategies concerning emotions. For example, successful problem solvers are prepared to tolerate frustration on their way towards solution.

The distinctions made here are important theoretically and methodologically and there are also implications for teaching practice. Relevant emotional education would include the awareness of cognitive emotions and knowledge of their importance to thinking skills, acceptance of emotions as part of cognitive processes, and finally, increased control over one's own cognition. (See also Hannula, 1999).

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THE ROLE OF TENDENCY TO FOCUS ON NUMEROSITIES IN THE DEVELOPMENT OF CARDINALITY

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The aim of this study is to explore, if the lack of spontaneous tendency to focus on the quantities of objects indicates slower development of cardinality in three-year-old children.

Cardinality grows out of children's experience with counting in a manner similar to that in which a symbolic concept of print grows out of children's experiences with the alphabet. Cardinality requires both explicit knowledge of the relation between numbers and quantity, and the attentional procedures for focusing on what is counted. (Bialystok & Codd, 1997.) In the area of expertise research, it has been established that the amount of deliberate practice as well as the age at which it is started, is related to the level of performance (Ericsson & Lehmann, 1996).

39 three-year-old children participated this study. The video recorded tasks were presented individually. In the first task, the child was asked to imitate the experimenter when the experimenter lifted carrots to a bunny. In the second, task the child was asked to bring different multi-legged caterpillars as many socks as they needed.

According to the analyses of videofilms, children who did not spontaneously focus on the numerosities in the tasks, were not as skilled in recognising and producing small quantities than those who immediately focused on the numerosity.

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RESEARCH TASKS IN A CONTEXT OF VOCATIONAL EDUCATION

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Gerard Seegers (Leiden University)

Mathematics education is gradually changing from teacher oriented learning towards student regulated learning within a co-operative learning context. This means that the learning context and also the didactics for teaching mathematics have been changed, implying that students have to plan, to discuss and construct their own knowledge. In our study a didactical model was used in which students were working co-operatively in small groups of four students, while the teacher coaches his students.

Within this context the mathematics curriculum for secondary education has been reformed. Two components of this reformed mathematics are the use of research tasks and the graphic calculator: research tasks are structured as problems where the use of a graphic calculator is necessary. The aim of these research tasks is to promote discussion between students when they start a new subject or to end a subject. After a research task is completed students have to present their results or to write a report about their findings.

In our study we observed students while working in small groups on the research tasks. While observing these groups we were focused on the interaction processes between of the students and the coaching of their teachers.

Effective problems allow a clear interpretation of the problem without suggesting or pre-structuring the solution process. Preliminary analysis of verbatim protocols and observation shows that students were highly motivated to work on these tasks. Observations also showed that it is difficult for the teachers to coach students while they are working on these problems.

On the basis of the observations and the preliminary analysis it turned out to be necessary to develop a coaching program for the teachers to improve their coaching behaviour. This means that the instruction to use a graphic calculator will be improved and the feedback how to work in co-operative groups. In the future the study will be directed towards the effects of improved coaching behaviour of the teachers on the learning processes and learning outcomes. Both the preliminary results of the protocol analysis and the coaching program for the teachers will be discussed.

SURPRISES IN INTEGRATING THE COMPUTER IN TEACHING MATHEMATICS

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Many agree that there should be greater integration of use of the computer in schools in general and in the study of mathematics in particular. The NCTM Standards (1989) and the Harari Committee Report (1992) already recommended this ten years ago.

In our efforts to integrate use of the computer in teaching mathematics in teachers training colleges, we have developed a "Computer-Oriented Numerical Math" course in which the computer is used as a tool for solving mathematical problems and as a tool for exploring ,discovering and illustrating abstract terms (Hoffmann 1996, 1999).

We believe that there are many advantages to students building the mathematical computer programs themselves, after exploring and discussing the various existing methods. During the next stage, the students 'run' the programs they have written, check the results obtained, discuss them and draw conclusions.

But sometimes at this stage, to the students' surprise, strange results are obtained, very far from those expected. We shall discuss these surprises in our lecture.

We shall deal mainly with:

- Surprises in the numerical computation of the numbers e and π.
- Surprises in solving linear systems.
- Surprises in solving quadratic equations.
- Surprises in performing basic algebraic calculations.

These surprising cases, serve as a source that motivates an increased interest in the study material, and willingness to explore and deal with the subject further. We are of the opinion that it is important to introduce these problems and deal with them. Immediately following this, we discuss the reasons that caused the incorrect results to be obtained, and propose various ways how they can be overcome.

Students learn that it is important to consider the results obtained critically even in a case where the program they have written and studied in fact 'ran' faultlessly. They are usually happy to discover (as one student put it) that "the computer can also err, and the computer is not 'all-powerful' ".

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A Survey on Problem Solving Instruction of Elementary School Mathematics from Constructivist Perspective in Japan

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The high achievement in mathematics and high quality of mathematics teaching in Japanese elementary school had caught the attention in many countries. Although we do not really feel of such high level and quality in our own culture, many comparative studies had provided strong evidence towards such tendency.

The purpose of this paper is to clarify the characteristics of problem solving instructions in Japanese elementary school mathematics especially from the constructivist perspective. After considering the theoretical background about problem solving instruction from constructivist perspective(Cobb et al.,1992 & Kamii, 1990), I would like to include the U.S.-side reports of the comparative study by Stigler et al. (1988, 1996) as an external viewing of the Japanese mathematics teaching. At the latter part in this paper, I will elaborate the Japanese elementary teachers' views in teaching the algorithm of two-column addition from the constructivist perspectives and illustrate a typical lesson of such algorithm in our textbooks.

Based on these considerations, we could agree that problem solving instructions based upon constructivism are being practiced widely in elementary school mathematics in Japan. There are three evidence in this paper to support such claim. First, the U.S.-side viewing on teaching mathematics in Japan had suggested that the predominance of Japanese mathematics education is to promote students' thinking in problem solving. Second, more than 60% of novice teachers as well as the majority of experienced teachers have the constructivist views in teaching the algorithm of two-column addition in the lesson despite the fact that textbooks illustration is not in the constructivist perspective. Third, experienced teachers are more strongly agreeable to the teaching elementary school mathematics based upon constructivism than the novice teachers. Although these evidence have reflected upon our success in teachers education especially the in-service training, I would like to propose further improvement of the description in the textbooks that I believe have a strong influence on the actual teaching in classroom.

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WHAT ARE THE CHARACTERISTICS OF THE PROBLEM SOLVING PROCESS ?

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1. Purpose

(1) What are the characteristics of the problem solving process, if students can solve problems in variety way?

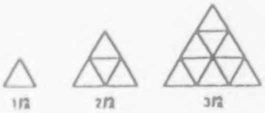
(2) Which solution method do students select as the best among several methods they approach and from which view point do they evaluate it?

(3) Do the students plan to improve the method they select?

2. Method

Six students in the 6th grade were given two problems and interviewed. The following is one of the problems used.

Using equilateral triangles of 1cm on each side, you can construct a triangle as below: When you construct a triangle of seven levels, how many triangles do you need?



3. Result

(1) Three types of problem solving process were identified.

Ⅰ Students began to solve it by a poor but easy method, then they tried to solve it by an improved method.

Ⅱ Students repeated the same solution method.

Ⅲ Students began to solve it firstly by using a better solution method, then they tried poorer method.

The characteristic of type Ⅰ appears to be an "improved" process, type Ⅱ a "revised expression" process and type Ⅲ a "change of viewpoint" or "no relationship" process.

(2) There are different reasons for the best method they select. For example, general solution, easy solution, easy to see and understand, interesting method.

(3) Some students do not evaluate the method of mathematical expression as the best way.

(4) Students do not have any plans of improving the best method they select, even if the best method is not improved mathematically.

GENDER-DIFFERENNCES RELATED IN ELEMENTARY SCHOOL STUDENTS' MATHEMATICS ANXIETY OF JAPAN

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Introduction:

The purposes of this study were to explore the general of gender-differences in high-grade elementary school students' mathematics anxiety by considering the following questions: Are there differences and gender-differences in affective characteristics toward mathematics between students with high mathematics anxiety and those with low mathematics anxiety ?

Procedures:

The Shimane-ACTM was used to measure four affective characteristics toward mathematics ; self concept toward mathematics (SE), achievement motivation in mathematics learning (MO), mathematics anxiety (AN), and attitudes toward mathematics (AT). The Shimane Affective Characteristics Test toward School Mathematics (the Shimane-ACTM) was developed by the author's investigation on the affective domain in mathematics learning for 16 years. (Ito, T., 1995)

The subjects of this study consisted of 250 fifth- and sixth-grade public school students (122 males and 128 females) in Matsue City, Shimane, Japan. They were conducted during the beginning of the spring term of the 1995 to 1997.

Results:

When with the gender and the level of mathematics anxiety 250 students were classified into 4 groups: M-L, F-L, M-H, and F-H group, it was obtained that there was mathematics anxiety (AN) of M-L group > that of F-L group > that of M-H group > that of F-H group in weak order of mathematics anxiety. Gender differences in mathematics anxiety and self concept toward mathematics were found between high mathematics anxiety students and low mathematics anxiety students .

Note.: > showed there was significant difference with mean difference test at 5 % .

Conclusions:

The results of this study seemed to suggest that mathematics anxiety for elementary school male students had the high interactive effect with self concept toward mathematics whereas mathematics anxiety for female students had the interactive effect not only with self concept toward mathematics but also achievement motivation in mathematics learning and attitudes toward mathematics.

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The Devolution of Cabri Activities: From Researchers to Teachers to Students

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In Brazil, the current curricula for early years mathematics propose that geometrical concepts should be presented through a variety of perspectives, linking theoretical and empirical investigations with everyday experiences. The problem is that most primary teachers, who received only a superficial education in geometry, do not know how to implement such an approach. In the collaborative research project *Space and Shape*, a group of teachers and researchers worked together to change the way geometry was viewed and taught in a São Paulo school. One question that was explored during this project was the role that the Cabri-Géomètre might play in this transformation.

In the early stages of the project, the idea was that we, the researchers, would devise activities so that the teachers could appropriate all the Cabri tools as well as extend their understanding of geometrical concepts by constructing and investigating objects created with these tools. Once teachers were confident in manipulating the software, we envisaged that they would modify our activities for use with their own students. In practice, this turned out to be less efficient than we had imagined. The teachers' concern from the beginning was classroom-use, whilst ours was on geometry content. This mismatch meant that the teachers never truly appropriated the activities as their own.

It was clear that a different approach was necessary. We decided classroom integration should be given a more central role and that each week the teachers, together with the researchers, would design two Cabri activities (one for students aged 6-8 years and one for 9-11 year-olds) to be implemented in their own classrooms. In the weekly meetings, the researcher no longer acted as the task-provider. Instead, their role was to make suggestions, demonstrating relevant Cabri possibilities, and to encourage the teachers to reflect on what happened when their design decisions were applied in the classroom. Over the 6-month period of meetings, we detected a number of changes in the types of activities that the teachers came up. These changes were illustrative of transitions in the ways these teachers viewed the teaching and learning of geometry.

We will present a sample of the teacher-designed activities to show: how emphasis on uniform responses evolved into the consideration of student-generated definitions; how tightly-specified worksheets were relaxed to encourage students to make decisions about problem-solving strategies; how replications of static paper and pencil tasks were replaced by tasks exploiting the dynamic processes afforded by Cabri; and how the software became a way to introduce and explore new geometrical ideas rather than just to illustrate those already studied in the paper and pencil context. In summary, the evolution of the activities is consistent with a move from a view of geometry learning as reproduction and memorisation to a view of geometry teaching as the devolution of learning situations to students.

The Formative Effect of Process-Oriented Assessment in the Mathematics Teachers' Practices

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A process-oriented assessment is essential for teachers to gain insight into the students' thinking processes, their mathematical understanding, competencies, and knowledge. To facilitate this way of assessment we carried out a pilot project in 1997/98 aiming at developing new assessment material inspired by the work of van den Heuvel-Panhuizen[1] and Neumann[2]. The material consists of written tasks, including simple calculations, open-ended questions and problem solving, and it intends to allow the students to explain their ways of solving a given task by writing and/or making drawings. In this way it offers a means for teachers to get feed back about their own teaching.

In 1998/99 we carried out an in-service education programme intending to familiarise teachers with current ideas about assessment. As part of the programme the material from the pilot phase was introduced and the teaching were encouraged to use it as a part of their teaching practices. We were interested in knowing if, when using the material, a) the teachers experienced new, unexpected insights into the students' knowledge and competencies, b) students' activities provoked the teachers to reflect differently on the teaching/learning processes, c) this way of assessment had had any impact on the teachers' planning of their activities.

We tried to identify the formative effect of this type of assessment on the teachers' practices by a qualitative analysis of questionnaires given to the teachers, and of the experiences gained during the in-service education programme. We did so using the model of teachers change presented by Clarke & Peter[3]. The conclusion was that the information achieved about the students' learning processes had promoted reflections and changes, or intentions of changes, in the classroom practices. This result suggests that the use of process-oriented assessment material increases the teachers' awareness on the teaching and learning processes and thus facilitates the teachers' possibilities of supporting the students' individual learning processes.

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THE CONTEXT IN THE DISCUSSION ACTIVITIES OF MATHEMATICS CLASSES

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The purpose of this study is to present the framework for considering the function of communication in the classroom communities and to analyze some episodes of the class discussions. For this purpose we focus on the multiple contexts that exist in discussion activities.

Cobb et. al. (1993) discussed two kinds of the teacher's utterances which are on "talking about and doing mathematics" and on "talking about talking about mathematics". The aim of the latter is to renegotiate classroom norms, which are social norms and sociomathematical norms. They are regarded as social contexts. In addition, there are mathematical practices (Cobb, 1999), which include contexts (Wood, 1999). These form the cultural space. With respect to the former, I present two different kinds of contexts: the communal contexts, in which some ideas for the goal are shared with the teacher and the students, and the participant's contexts, which are individual. Within such a framework, I have researched the features of the interactive discussion activities in two different cases. The first case is related to forming the communal context collaboratively. The second case concerns presenting the participant's own context with regard to the communal context. These contexts exist in the cultural space as a mathematics class and the interaction of mathematical ideas works effectively.

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INCONSISTENCIES ON THE CONCEPT OF GEOMETRY

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The phenomena of consistencies on mathematical concept have been investigated in detail. But further investigation is necessary in the field of geometry. The purpose of this research is to clarify the characteristics of inconsistencies on the concept of geometry.

The term "inconsistency" is used as the situation that if p is a proposition, then both p and ~p hold simultaneously. And the inconsistencies on the concept of geometry can be classified into two kinds of types.
One kind of type is classified from a viewpoint of objects of inconsistencies.
External inconsistency: inconsistency between the student's concept of geometry and mathematical concept of geometry.
Internal inconsistency: inconsistency within the student's concept of geometry.
The other kind of type is classified from a viewpoint of student's consciousness.
Explicit inconsistency: inconsistency that is noticed by student.
Implicit inconsistency: inconsistency that is not noticed by student.
External inconsistency has been called "conflict " or "misconception" and explicit inconsistency has been called "cognitive conflict" or "disequilibrium" .

Observers can easily find out student holding external inconsistency because mathematical validity of the student's concept of geometry is judged on the basis of absolute mathematical concept of geometry. On the other hand observers can't always find out internal inconsistency which is generated from the difference between verbal representation and imaginary representation in student's individual concept of geometry.

As far as I investigated about the recognition of trapezoid, 71% of undergraduates showed some evidences of external inconsistency and 82% of them showed evidences of internal inconsistency. For example some undergraduates wrote a correct definition of trapezoid by linguistic representation, but couldn't identify figures of square, rectangle and so on as trapezoid because their shape were not looked like trapezoid by imaginary representation. The results would appear to suggest that many undergraduates hold inconsistencies on the concept of geometry because of the effect of imaginary representation.

Other undergraduates who wrote definition of trapezoid: "a quadrilateral with only one pair of opposite sides that are parallel" could identify the figures having only one pair of parallel sides as trapezoid. They showed evidences of external inconsistency in all problems, but their answers were all consistent with one another. These students hold both external inconsistency and internal consistency.

What is significant in the teaching and learning of geometry in the light of inconsistencies is to make student become aware of the inconsistencies that is to exchange from implicit inconsistencies to explicit inconsistencies.

IMPROVING MATHEMATICAL REASONING:
THE ROLE OF MULTILEVEL-METACOGNITIVE TRAINING

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For more than a decade, research in the area of mathematics education has looked for instructional methods that have the potential to enhance mathematical reasoning. The present study investigates mathematical reasoning under three settings: (a) cooperative setting embedded within Multilevel-Metacognitive Training (implemented in mathematics and English classrooms); (b) a cooperative setting embedded within Unilevel-Metacognitive Training (implemented only in mathematics classrooms); and (c) a cooperative setting with no metacognitive training. The metacognitive training was based on the IMPROVE method (Mevarech & Kramarski, 1997) which emphasizes reflective discourse in small groups by providing each student with the opportunity to be involved in mathematics reasoning via the use of metacognitive questions that focus on: (a) the nature of the problem/task (b) the construction of relationships between previous and new knowledge; and (c) the use of strategies appropriate for solving the problem/task.

We hypothesized that providing metacognitive training in both mathematics and English classrooms would exert more positive effects on students' mathematical reasoning than implementing metacognitive training in only mathematics classrooms. The hypothesize is based on the assumption that students who are exposed to multilevel metacognitive training may be able to generalize the use of metacognitive processes beyond a specific domain.

Participants were 182 Israeli students who studied in six seventh grade classes. Results indicated that students who were exposed to the multilevel metacongitive training significantly outperformed their counterparts who were exposed to the unilevel metacognitive training, who in turn significantly outperformed the control group. The effects of the multilevel metacognitive method were observed on students' mathematical reasoning, students' mathematical explanations manifest during the solution of mathematical problems, the solution of outhentic task, and on the activation of metacognitive processes while solving mathematical problems. The theoretical and practical implications of the study will be discussed.

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CHANGING TEACHERS ASSESSMENT PRACTICE THROUGH AN IN-SERVICE PROGRAMME

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Curriculum 2005, South Africa's version of what is commonly known as Outcomes Based Education(OBE) has been introduced in South African Schools since 1996. The training programme followed by teachers implementing Curriculum 2005 has been rigorous, intensive and philosophical. As one of the trainers for Curriculum 2005, I have made many observations about the training process and subsequently become very sensitive to the needs and "cries" of the classroom teachers.

Implicit within this curriculum model is the crucial component of assessment. Many teachers confess that introducing Curriculum 2005 is manageable, but developing OBE assessment practices seems to haunt and elude many teachers. My earlier research(Krupanandan ,1999) around assessment practices of teachers, revealed that teachers were in the first place underprepared , and secondly many teachers have fallen foul to moving their assessment practices to areas were they were more comfortable, rather than conducting their assessments in the precise areas where teaching and learning outcomes have been defined. The research also concluded, that there is still a strong leaning by teachers, in using assessment strategies that were used in the past and therefore no change in practice has taken place.

In 1999 I included studies around Curriculum 2005, and more deliberately assessment in my in-service course offered by the teacher training college where I am employed. These were to mathematics teachers who had enrolled with the college for a Further Diploma in Education, and were working with Curriculum 2005 daily in their classrooms. I drew up a programme for the year, including 8 contact, face to face sessions with the teachers. These sessions were designed to provide hands on experience to practicing teachers, to develop and design assessment practices that are consistent with an OBE curriculum model. Visits to schools or sites of learning was also done, before and after the course.

This paper will provide the results of this research, in an attempt to meet the needs of teachers in the critical area of assessment , and provide confidence to teachers to implement a new curriculum model, that presents many challenges in practice.

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IMPORTANCE AND DIFFICULTY OF SCHEMA INDUCTION IN ANALOGICAL PROBLEM SOLVING

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The purpose of this presentation is to illustrate importance and difficulty of schema induction that is essential in mathematical problem solving instruction by analogy.

In school mathematics textbooks, we find a common arrangement of problems, where one typical problem (example problem) and its solution are presented, then one or two problems (application problem) which can be solved using the same method as one of the example problems will be posed. In this plan, teachers expect students to learn mathematical methods to solve a class of problems and to apply them to new problems. The reasoning which students use in applying a prior solution to a new problem could be viewed as analogical reasoning. In this type of lesson, it seems to be easy for students to solve the following problems because they are very similar to the example. However, as we well know, many students have difficulty solving problems on a later test in which there is no example problem. In that case, students have to retrieve a potential solution from various types of examples.

I will focus on critical roles of an abstract schema in analogical problem solving and some conditions to facilitate schema induction. Cognitive studies concerning analogical reasoning suggest that an abstract schema is important in noticing a similarity between two analogs and in facilitating analogical transfer. An abstract schema is important in learning and teaching of mathematical problem solving, too. However given a practical situation, we face difficulty making an abstract schema. Using some examples from school textbook, I will illustrate that it may be impossible to formulate an abstract schema among various problems which have different semantic situations but a common solution.

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Representational Structure of Numbers in Mental Addition

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Hajime Yoshida, Miyazaki University, Japan

Resnick(1983) assumed that a representation of numbers in preschool children could be characterized as a mental number line in which numbers were linked to each other by the next relationship. However, Kuriyama and Yoshida(1988) suggested that preschool children represented numbers to 5 as a firm structure or privileged anchor for the numbers below 10. The purpose of this research was to investigate whether or not elementary school children have the representational structure of the number 5 by analyzing the reaction times in mental addition tasks.

Method: *Subjects.* The subjects were 36 first graders and 34 fourth graders attending a public school. *Materials and Procedure.* Two-term addition problems were used in this experiment. Both the augend and addend were single digits. All response recording were accomplished by a laboratory microcomputer.

Results and Discussion: The data on reaction times showed that the first-grade children solved the problems with 5 in either of the two addends faster than the problems without 5 in both addends. This suggests that school children represents the number 5 as a privileged anchor. Further, fourth graders did not represent the number 5 as a privileged anchor. This might suggest that by learning the decade structure on numbers for four years, school children integrated the number 5 as a privileged anchor to the decade structure. It was proposed a new model of number representation.

SOUTH AFRICAN MATHEMATICS TEACHERS IN TRANSITION. -MAKING SENSE OF THE MATHEMATICS SPECIFIC OUTCOMES.

Agatha Lebethe & Gabeba Agherdien , School Development Unit, University of Cape Town, South Africa.

The implementation of an Outcomes-based education system in South Africa has brought a wave of spontaneous debate among the South African schooling community and abroad. Since its implementation in 1998, research has shown teachers have different understanding of OBE, display uncertainty and insecurity about their practice irrespective of the aggregate levels of institutional resources or years of personal experience. (Jansen, 208).

As inservice providers we have been inundated with desperate pleas from teachers to help them make sense of the outcomes, to understand the OBE jargon and to make the "maths outcomes real" in their classrooms. The research became an urgent need to assist us in our attempt to help these teachers and begin to understand mathematics teachers in transition. It is also the intention of the study to contribute to the conversations around the effect of the implementation of such a complex system such as OBE on teachers who received minimum formal preparation and training and very little change to the material resource base to enable the new curriculum. We see the research as ongoing and necessary to demonstrate the distance between policy and practice, between the intentions of Government and what teachers experience and so contribute to the question posed by a noted South African Academic, Jonathan D Jansen, 'What does the South African experience tell about the classroom practice.

This paper will show the results of a research conducted among teachers described as 'key' teachers by an inservice project because of their dedication and involvement to inservice, they are seen as initiators in their schools and should have a sound understanding the Mathematics Specific Outcomes. The findings of this study are based on an analysis of audiotapes, interviews and questionnaires with participants.

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ON THE TRANSITION BETWEEN SPACE AND PLANE*:

A.L. Mesquita (U. de Lille/IUFM)

The study presented here concerns a longitudinal project on the utilization of tridimensional geometry at school. Its aim is the valorization of geometry and space at school (valorization which is claimed as a main goal of the teaching at school level, cf. for instance Mammana & Villani, 1998) and it is being implemented in a primary school of the northern of France since 1997/1998, with about fifty pupils during their school attendance (from 6 to 11 years-old). We assume that a priority should be given to space in the beginning of the learning of geometry; also, the construction of objects and other activities of material and symbolic manipulation, in the sense of Caron-Pargue (1981), have an important role in our approach; finally, a special attention is given to the different semiotic systems of presentation of knowledge used in geometry and to their articulation (Duval, 1999). In some previous PME conferences we report other phases of this study (cf. for instance Mesquita et al., 1999).

This presentation reports a particular moment of the study, the transition (in both senses) between space and plane, with 7 to 8 years-old pupils. The utilization "on situation" of the geometrical notion of orthogonal projection is a crucial point in this phase of our study. In this presentation we will show how, by the utilization of specific structured problematization activities (Mesquita *et al.*, 1999), pupils used this notion, enabling such a transition between 3D- and 2D-points. We will present some of these activities, focusing on the way of how pupils used this notion, as a way to pass from a 3D-point to its projection in the plane, and to associate 3D-points to a 2D-projection.

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* Funding from IUFMNPdC, project # R/RIU/98/079. Other participants in the project are: Anna Abbes, Francis Delboe, Judith François, Nathalie Owsinski, Annie Régnier, Sabine Rossini, Jean Vandenbossche, Nathalie Vasseur.

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Students' conceptual change of the equal symbol:

On the basis of the notion of epistemological obstacle

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The notion of epistemological obstacle(EO) is indispensable to studying for identifying learners' difficulties encountered in their learning processes, and to deciding more appropriate strategies for teaching. In the historical development of mathematical concepts or knowledge, many EOs have been overcome. This means that we have to overcome the EOs for developing our concepts or knowledge in the learning of mathematics.

On the other hand, we need to study how to describe the processes of students' overcoming EOs, the condition for students to overcome EOs, and the significance which a learner experiences the processes of overcoming EOs.

We need to distinguish three categories for description of students' overcoming processes; *notion*, *event*, and *conviction*. The category *notion* includes students' ambiguous ideas, images, and mental models. The category *event* means students' concrete experiences which *notion* is laden. The category *conviction* means students' attitudes towards mathematical knowledge. By shifting *conviction*, the student can change his/her knowing as a whole, that is, the student can overcome his/her EO. Using these categories, four states of learners confronting EOs are described: 1) persistence in subjective facility; 2) justification as social adaptation of *event*; 3) becoming aware of an EO; and 4) overcoming an EO. Comparing these four states, the following viewpoints are inferred for identifying the learner's way of concerning with EO: social context; reflection towards one's knowing, and the *conviction* -shift. If we do not interpret their learning of mathematics with these viewpoints, we could not understand the very process which a learner overcomes an EO. Hence, such viewpoints are identified as the significance of the EO.

On the basis of the above theoretical framework, we discuss students' conceptual change of the equal symbol in school mathematics, and construct a framework for it as follows: 1) evolution of the definition of the equal symbol; 2) change of the objects connected by the equal symbol; 3) a student's "equality' which evaluates the equal symbol.

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CHILDRENS' STRATEGIES TO CALCULATE PRICES OF FRUIT WITHIN A REAL WORLD SETTING

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One of the focal points within problem solving research is the impact of real world settings and what skill children require to find respective answers. Since money plays an important role in early arithmetic applications (e.g. prices of goods) there is a need to explore the childrens' strategies in recognizing the relationship between the prices and the weight of goods. This investigation into the "economic world" of elementary school children proves to be essential of social constructivism, functional thinking and the existence of subjective experiences.

Research questions

What are the students' reasoning strategies in calculating prices for different amounts of fruit? How does the concept of proportionalty develop?

Method

The sample comprises 99 students grade 1 to grade 4. Each student was individually interviewed while he or she passed by four tables resembling fruit stalls in a market. Each table carried a different sort of fruit (grapes, pears, apples and plums). They varied in quantity (1 kilogram and more), they showed different sizes and colours (green and blue grapes, green and red apples) and were marked with different prices (e.g. 1 kilogram 3 DM, 1 kilogram 1,50 DM). For each of the four sorts of fruit the students were asked about the prices of the different portions. They could confirm the weight of the apples by using the scale.

Results

The answers reveal that first and second graders use criteria like taste, colour, size, amount and usefulness to reason for their prices. The most frequently used arguments are the amount and the size of the fruit pieces. Over half of the third and forth graders use porportionality. One third of them are able to calculate with prices like 1,50 DM and are able to find double as much with an additive or multiplicative strategy.

TEACHERS' POSITIONS IN ASSESSMENT DISCOURSES: INCLUDING A SOCIOLOGICAL PERSPECTIVE ON THE MATHEMATICS CLASSROOM

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The study of assessment at PME has traditionally focused on what assessment can reveal about students' mathematical understanding. More recently there has been a turn to look at teachers' beliefs and practices. Morgan (1996) identified the importance of recognising the interpretative nature of teachers' evaluations of their students' performance and the ways these evaluations are influenced by the resources teachers bring to the assessment situation. Empirical study of assessment practices indicated that these resources were drawn from different, sometimes contradictory, discourses and that the various ways teachers were positioned within these discourses could lead to different evaluations of the same student texts (Morgan, 1994; Morgan, 1998). This raises important questions about what these discourses are and what positions, evaluation criteria, and orientations towards the practice of assessment are available within them. We shall argue that a sociological perspective, drawing on the work of Bernstein (e.g. 1996), can illuminate these questions and enrich our understanding of teachers' assessment practices.

From the point of view of Bernstein's work, we are concerned with how official discourses of assessment of mathematical performance are recontextualised at the level of school. We can ask whether teachers accept or reject the criteria and procedures of the official discourse, in which case they tend to speak the voice of the legitimate text, or the voice of other texts. And we can ask: what resources do they draw on to recontextualise the dominant discourse? We can say that they draw either on recontextualisations of the specialised discourses produced at the field of pedagogic knowledge production, such as the mathematics education research community, or on non-specialised resources, i.e. everyday experiences. Accordingly, the orientations of the teachers are either to the students' texts or to the student himself or herself. We will illustrate these positions by drawing on texts of interviews with teachers.

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Teaching and Learning Statistics:

Diagnostic and Support Materials for Teachers and Students.

James Nicholson & Gerry Mulhern

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Recently, Batanero and Truran (1998) highlighted the need for all adults to make some use of 'advanced statistical thinking' in the form of making inferences from data. Hawkins (1989) and reported a survey showing that a substantial proportion of teachers surveyed in the UK were not well equipped to teach the statistics which was demanded of them. Although some in-service training has taken place since then, the volume and level of sophistication required have also increased dramatically, and the situation is arguably more acute now.

We are currently engaged in a project to help teachers devise appropriate classroom strategies for diagnosing common conceptual difficulties among their students, and effective methods for overcoming such difficulties.

We will present examples of the materials developed in two areas: Sampling Methods and Correlation & Linear Regression, and report the outcomes of the initial trialling in schools. These attempt to address some of the key issues identified in interviews with teachers and examiners. For example:

- that students have difficulty in assessing the reliability of predictions made using a line of regression.
- That students find it difficult to identify the contexts in which different sampling methods are advantageous because they have little or no experience of using them.
- that students find it difficult to apply abstract principles to concrete scenarios.

This work is supported by a grant from the Nuffield Foundation.

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LEARNING AND TEACHING THROUGH PROBLEM-SOLVING: REFLECTING ON TWO SPATIAL CASES[1]

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Effective school teaching enables learners to perform relevant tasks of learning. To this end learning conditions that are conducive to learning need to be provided in classroom situations. From a constructivist-based social cognitive view meaningful learning is defined as the goal-oriented, active, constructive, cumulative and self-regulated processing of information into useful knowledge. Moreover, social cognitive scientists posit not only that learning can best be characterized as problem-solving, but also that teaching itself resembles problem-solving. In addition, classroom ecological research suggests that the context of teaching and learning in particular is an essential determining factor regarding the success of classroom events. Hence, the utilization of problem-solving in a relevant context seems to be an essential aspect of effective school teaching and learning, particularly of spatial concepts.

The purpose of the paper is to reflect on the possibilities of problem-solving in a relevant context with reference to effective teaching and learning of spatial concepts in school classrooms. The analysis draws on two recent classroom experiments concerning meaningful acquisition of spatial concepts, conducted in multicultural South African classes. One comprised developmental research with Grade 1 learners following an experimental problem-centred learning program, based on the Van Hiele Theory. The other involved Grade 7 learners following the prescribed school syllabus with two experimental groups following a Van Hiele-based problem-centred program, while two control groups followed a conventional program. The combined results of the two cases:

- Emphasize the importance of language in the learning of spatial concepts;
- Reveal that young learners employ specific authentic strategies, some of which are quite sophisticated, to resolve spatial problem situations;
- Suggest that problem-solving in a co-operative social context renders pertinent advantages for learners, particularly in respect of their level of geometry thinking and conceptualization, concentration and motivation (at practically significant levels, using effect sizes);
- Suggest that such an approach impacts on school teachers' thinking about classroom teaching and learning in a fundamental way, provided they are open and committed to experiment with it in supportive working conditions.

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[1] The two cases form part of the Spatial Orientation and Spatial Insight (SOSI) Project, sponsored by the National Research Foundation (NRF, formerly FRD) of South Africa

THE IMPACT OF A VIDEO CLASS SYSTEM ON THE TEACHING AND LEARNING OF JUNIOR SECONDARY SCHOOL MATHEMATICS

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In outcomes-based education, as currently being implemented in South African schools, the outcomes are focused on the learner, providing guidelines regarding competence in respect of knowledge, skills and dispositions to be attained (Brodie, 1998). This implies learner-centred teaching in mathematics classes (Department of Education (DoE), 1997). Learners need to take responsibility for and make appropriate decisions regarding their own learning; they have to be independent in their learning and thinking processes, effective in their own learning activities; and able to apply self-assessment procedures (DoE, 1998).

There are though obstacles in the effective implementation of a learner-centred approach, e.g. the cost factor, rationalisation, shortage of aptly qualified mathematics teachers and large classes. Hence, it is necessary to investigate workable ways to overcome these restrictions. The main objective of the research was to investigate the VCS as a supportive way to cost effective teaching, as well as to determine the influence of the VCS on the teaching and learning of school mathematics.

Two secondary schools where the needed facilities could be provided, participated in the project. At one school mathematics teachers for grade 8 who were willing to participate, were involved. One grade 8 class (E1) learned mathematics while the lesson was being video-taped. A second class (E2) learned mathematics through the video-taped lesson without the constant presence of a teacher in class; and a third class acted as a control group, continuing in the "normal" way. The teacher of the experimental classes visited the "video class" (E2) from time to time to supervise and conduct relevant tutorials. This teacher was trained in appropriate methods of mathematics teaching. The same procedure was followed at the other school, using grade 9 learners as the grade 8 learners were already involved in another mathematics project.

A pretest-posttest experimental design was employed, using as dependent variables the fields of the LASSI-High School Questionnaire (Weinstein & Palmer, 1990), adapted for mathematics, and the learners' marks for mathematics. A self-constructed observation schedule was used to qualitatively analyse events in the experimental classrooms.

Teachers used the VCS to reflect on their own teaching, while some learners used the tapes to catch up with mathematics done while they were absent. In one school practically significant differences (effect sizes) occurred between the three groups' mathematics achievement. This could be attributed to the teacher adapting appropriate teaching strategies, as well as the learners in E2 taking responsibility for their own learning. In the other school learners in the "video class" perform as well as the others. The results support Lowry and Thorkildsen's observation (1991) that a VCS does not have a negative effect on the teaching and learning of mathematics, and that of Huge (1990) that video recordings of mathematics lessons can successfully compensate for absent teachers. The results suggest that a VCS can contribute toward solving problems due to rationalisation, for absent or ill-trained mathematics teachers and large classes.

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TWO THINKING STRATEGIES IN WORD PROBLEM REPRESENTATION

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History gives numerous proofs of the link between two semiotic systems – the word and the image – and their important function in a student's progress in gaining further concepts, processes and understanding in mathematics.

When a teacher/researcher marks students' activities, he/she perceives only one of the two sides of a mathematical activity - the visible conclusion of the mathematical objectives and the valid processes used to solve a given problem. He/she does not see directly the hidden and crucial side (Duval, 1999, p. 24), namely the student's thinking, which can only be inferred/estimated and in many cases is the most important activity. The student's thought processes can be inferred from written work. In this contribution we focus on graphical representations of a word problem structure. We will study the spontaneous usage of geometrical/non-geometrical figures (Kubínová, Novotná, 1999). We will concentrate on two thinking strategies – 'advance organisation' and 'selective attention' (O'Malley, Chamot, 1990).

Sample. 12-14 year-old students from Prague and České Budějovice. All classes were non-specialised ones.

Method. Analysis of written work. Each student received one A4-sheet with the assignment written in the upper part of the sheet and was asked to put all calculations, figures, schemes, notes etc. on this paper.

Background. The word problem dealt with, consisted of two sub-problems, one having a pure multiplicative structure, the other a mixed additive-multiplicative structure.

Classification of the metacognitive strategies. The following criteria were used: *Advance organization*: congruent/non-congruent (Duval, 1999), processual/conceptual (Kubínová, Novotná, 1999), complete/non-complete record (Novotná, 1999); *Selective attention*: recorded information chosen by the solver, use of letters and other supporting symbols (Novotná, 1999). Students' original pictorial records will illustrate the results of the classification.

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Acknowledgement The research was supported by the projects GAČR No 406/99/1696 and by the Research Project Cultivation of mathematical thinking and education in European culture

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The Research of Planning in Open-Approach Method

- Through the Description by Opportunistic Planning Model -

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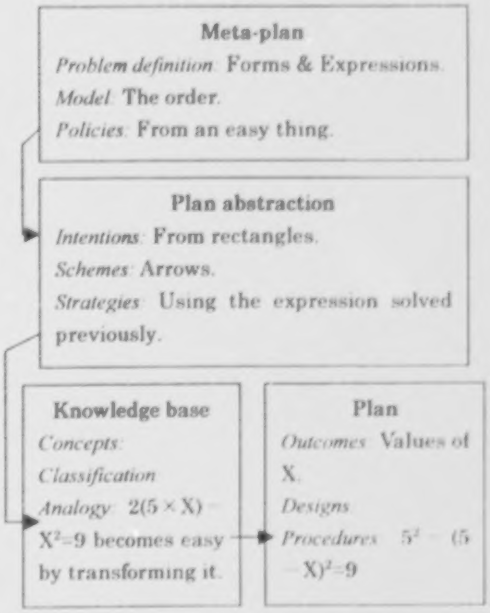
Purpose

There are two purposes on this research. One is to find out a suitable model to describe planning in *Open-Approach* (Nohda,1995)[1]. The conclusion is that *Opportunistic Planning Model* (Hayes-Roth, 1979)[2] is a suitable model. OPM is the abbreviation of Opportunistic Planning Model. There are four reasons to attain conclusion. ①OPM can describe various reasoning. ② OPM can describe partial planning. ③ OPM can describe policies and intentions of the planner that are driving-force in planning. ④ OPM can be made a change and become suitable for Open-Approach.

The other is to propose *the functions of planning* in Open-Approach. These are planning as a means of problem solving, planning as a means of creation and planning as configuration of reasoning.


Meta-plan
Problem definition: Forms & Expressions.
Model: The order.
Policies: From an easy thing.
Plan abstraction
Intentions: From rectangles.
Schemes: Arrows.
Strategies: Using the expression solved previously.
Knowledge base
Concepts:
Classification
Analogy: 2(5 × X) − X²=9 becomes easy by transforming it.
Plan
Outcomes: Values of X.
Designs
Procedures: 5² − (5 − X)²=9

Case Study

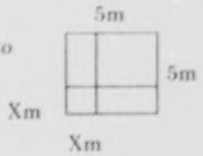
I questioned a boy who was a second-year high school student.

Situation: We would like to divide square flower bed which area is 25 ㎡ into four parts like a figure and plant the salvia which area is 9 ㎡.

Problem 1: Let's draw various forms for planting it!

Problem 2: Let's make various equations!

Problem 3: Let's solve equations!



OPM can show various reasoning. For example, it can show causal reasoning as the arrows from *strategies* to *procedures*. Also it can show partial planning. Though the planner does planning on the basis of *policies* as a whole, he is dependent on *strategies* while some parts of planning. It is more important than above reasons that OPM can describe driving-force in planning. With OPM, we can notice planning in meta-level through some descriptions that are "From an easy thing" in *policies*, "From rectangles" in *intentions* and "Using the expression solved previously" in *strategies*.

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LEARNING GUIDANCE FOR DEVELOPING CREATIVE ABILITIES AND ATTITUDE
-APPLYING THE OPEN APPROACH METHOD IN THE MATHEMATICAL LEARNING PROCESS-

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Shinji Hirotani, Takamatsu Agricultural High School.Japan
Hiroshi Sakata, Okayama University.Japan

In the process of teaching mathematics,it has been considered important to encourage children to have mathematical problems and concepts related to their own lives and to give them an interest in creating solutions. Therefore questions which allow children to try various ways of solving a problem and different approaches as well as learning new ways to apply mathematics are needed. Moreover the opportunily for children to create various solutions and to develop their mathematical thinking abilities through communicating with other children is also necessary. We believe that the Open Approoach Method developed by N.Nohda from research by S.Shimada is useful and effective.

We developed the following question according to the Open Approach Method. A variety of approaches are possible according to the individual's ability and age.

question : The following numbers are arranged in a certain order. What number comes next? Why?

2, 3, 5, 8, □ · · · · · ·

Students in classes from grade 7 to grade 12 tackled the problem enthusiastically and figured out 2 to 4 different answers for each. Tackling the problem individually,students shared a lot of ways of mathematical thinking through communicating with other students. Here are some of the answers the students come up with.(We use this equation to describe the student's explanations briefly.)

(1) 2,3,5,8,$\boxed{12}$,17,23,30··· $a_{n+1}-a_n=n$ (n=1,2,3)

Follwing difference progression goes to arithmetic progression. 90% of students came up with this answer.

(2) 2,3,5,8,$\boxed{13}$,21,34,55··· Fivonacci Progression, $a_{n+2}=a_{n+1}+a_n$ or $a_{n+2}-a_n=a_{n+1}$

This answer is found by 85% of the students.

Examples (3) to (6) were found by approximately 15% of the students

(3) 2,3,5,8,$\boxed{9}$,11,14,15··· ①$a_{n+3}=a_{n+6}$ ②$b_n=a_{n+1}-a_n$ 1,2,3,1,2,3 repeating series

(4) 2,3,5,8,$\boxed{2}$,3,5,8,2··· This is a cyclic progression. $a_{n+4}=a_n$

The clue of this progression is an analogue watch.

(5) 2,3,5,8,$\boxed{5}$,3,2,3,5,8,··· $a_{n+6}=a_n$ This equation describe a sine curve.

(6) 2,3,5,8,$\boxed{12}$,18,27,41···

○There is one 2 in 2. $2\times2-1=3$ ○There is one 2 in 3. $3\times2-1=5$

○There are two 2's in 5. $5\times2-2=8$ ○There are four 2's in 8. $8\times2-4=12$

This can be expressed by the following equation : $a_{n+1}=2a_n-[a_n/2]$ (n=Gauss)

Some other mathematically interesting answeres have been omitled because of space.

Since students were encouraged to figure out rules,expressions and various answers in their own ways,they worked on the questions eagerly.

They enjoyed the creative and mathematical activity.

Their mathematical view point has been expanded by directing the students to solve questions in individual ways and to exchange their ideas with each ather.

We believe we can develope a student's creative thinking ability and attitude toward mathematics by introducing the Open Approach Method.

In conclusions,here is a proberb which describes the core message of learning guidance. "I hear,and I forget.:I see,and I remember.:I do,and I understand.:"

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Geometric Construction as a Threshold of Proof: The Figure as a Cognitive Tool for Justification

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It is a global problem in mathematics education to understand how empirical knowledge of elementary mathematics is transformed to theoretical knowledge in the secondary level. In Japan, geometric construction is taught in 7th grade and logical proof in 8th grade under geometry. We consider geometric construction an intermediate step towards proof (Mariotti & Bartolini Bussi, 1998). However, most teachers tend to emphasize only its procedural aspect.

In this paper, we attempt to specify some conditions for students to sift from construction to proof in geometry. We observed and analyzed several 7th grade classroom lessons on geometric construction in which students were put in the context of "justification". The main topics were to draw an angle bisector and a perpendicular line by using various instruments and to justify the procedure.

We found that most students utilized another geometric figure (e.g. rhombus, isosceles triangle) as a cognitive tool in constructing a figure and justifying the procedure. Therefore our analysis is focused on the thought processes as shown in the geometric figures that the students expressed. The concept of image schema developed as a meaning-making function by Dörfler (1991) works well for our analysis. Especially its categories: figurative, operative, and relational can be used to make qualities of students' cognitive tools clear.

In the end, the following points are suggested.

- The geometric figure (e.g. rhombus) as a cognitive tool makes it possible for students to see another figure (e.g. angle bisector) in the figure, to read the theoretical relations in and between figures, to give them a logical form, and to estimate failure or success in their justification of construction.
- Factors for the student to succeed in the justification include; (1) externalization of the geometric figure imagined, (2) reflection of steps in the construction as conditions for determining the figure, and (3) reasoning based on operative image schema.

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EFFECTS OF BACKGROUND AND SCHOOL FACTORS ON BELIEFS AND ATTITUDES TOWARD MATHEMATICS

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The realization that mathematical skills are important to economic progress has prompted many nations to investigate the validity of their curricula in mathematics. TIMSS had aim the measurement of student achievement in mathematics and science and the assessment of certain factors influencing student learning. This study will examine predictors of beliefs and attitudes toward mathematics, focusing on those related to school and family. The data were collected in 1995. Altogether, 5852 students (Cyprus) participated in the study. The paths from educational background to SES, to attitudes, to beliefs, and climate were significant. The paths from reinforcement to attitudes, to belief for success in mathematics, and teaching were also significant, as were the paths from climate to teaching, the path from teaching to attitudes, and the path from SES to climate. Measures of fit included chi-square, GFI, AGFI, CFI and RMSEA. The results of this study indicated that two exogenous factors-educational background of the family, and student reinforcements-define a second order factor structure which includes endogenous predictors, socioeconomic status of the family, student attitudes toward mathematics, beliefs regarding success in mathematics, the kind of the teaching and the school climate.

GEOMETRIC DYNAMIC ENVIRONMENT AS A MEDIATOR AGENT FOR LEARNING GEOMETRY

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Under the perspective of Activity Theory, the mathematics classroom can be viewed as a system of activity, where students interact with each other and with the teacher, using mediational agents in their action (Wertsch, 1991; Engeström, 1998). In that system of activity, knowledge is shared among its elements, and mathematical meanings are constructed and negotiated (Meira, 1996). If the students are working in geometry, geometric dynamic environments such as *Sketchpad* can be a potential agent in that system of activity and a resource for learning (Laborde, 1998).

Drawing on a qualitative approach, we focus our ongoing research project on the mathematical activity of two middle classes, working in geometry with *Geometer's Sketchpad*. Lessons were observed and video-recorded. Students' group work was saved to be analysed and interviews were carried on.

From the analysis of data there is evidence to suggest that: i) when students use geometric dynamic environments (*GDE*) to think on geometrical objects and properties their activity is mediated in particular ways by those tools; ii) the exploration of the constructions that students make are more rich and purposeful if they are guided by the teacher; iii) geometrical meanings are constructed in the relation between the *GDE* used in the action, the tasks proposed and the conceptual framework.

The aim of this paper is to present and discuss these preliminary results.

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Reflections about the work with teachers

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In the last two decades the work with practicing teachers, future teachers, and my own work have lead to reflecting about our work as teachers. A teacher journey includes enthusiasm and problems as well as the development of professional knowledge fuelled with the discoveries and complexities of the teaching practice. We, mathematics teachers and mathematics educators around the world, feel many times a sense of being lost. It's a feeling that we do not know: a) what is going on in the classroom; b) why students are having trouble to grasp the content; c) how to incorporate innovations in the mathematics curriculum; d) how to accept the idea that we should change our teaching practice; and e) how to make alternative moves. In many situations we are stimulated and even pressed to think that we need to change our beliefs, attitudes, and implicit theories about mathematics and its pedagogy as well as our teaching and thinking about knowledge acquisition. In this work, I talk about a path followed towards becoming more experienced into acquiring knowledge about mathematics teaching and teaching education (Olson, 1997; Santos & Nasser, 1995). Through this path, theory can or cannot shed some lights on our understanding of math classroom, as well as may suggest ways to cope with teaching dilemmas. This is an interpretive study of the collaborative partnership experienced by a mathematics educator and a group of teachers. It pursues the following questions. a) How can teachers' knowledge and professional experience be taken into account when teachers are learning to become teacher-researchers? b)What forms of collaboration between a mathematics educator who acted as a mentor and supporter of teachers' work stimulate, enhance, block and/or disturb teachers' autonomy and learning process? Data includes written documents produced by the teachers, field notes from meetings and interviews. Exemplars of teachers' insights, reflections and actions show how evolved teachers' thinking about their teaching, the collaborative experiences and the interplay of theory and practice (Lester, 1996, Santos & Nasser, 1995). The study stresses the importance for teachers of working collaboratively towards the goal of becoming more knowledgeable in a community in which mutual respect, trust, collegiality, and concern are present.

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UNDERSTANDING HOW CHILDREN SOLVE COMPARE PROBLEMS

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Research has found that comparison problems are typically more difficult than change problems. It is not yet clear, however, how previous experiences and logical, linguistic, and representational factors interact, facilitating or hindering children's solutions to compare problems. In traditional mathematics classrooms and textbooks, combination and change problems are widely explored while compare problems are much rarer, thus depriving children of experiences that would allow them to deal with comparative statements flexibly. Students are often told that the keyword "more" suggests joining, adding, or buying, and "less" means taking away, eating, or losing. In this exploratory study we look in detail at how 12 third-grade American children interpret, represent, and solve compare problems. Analysis of the video-taped interviews, verbatim transcripts, and children's written productions will facilitate deeper understanding of children's multiple interpretations of compare problems and of the tension they experience in considering comparative terms as information about changes in quantities versus comparison statements.

Children were interviewed after participating in a series of five two-hour weekly meetings that were part of a broader year-long classroom intervention study (Carraher, Schliemann, & Brizuela, 1999). In the first two meetings, when given combine, change, and compare problems, the children overwhelmingly used addition to solve problems with the word "more" and subtraction for those with the word "less", thus providing wrong answers to the compare problems. During the following three weeks the children dealt with comparison statements, using "arrows" to represent and compare different amounts of discrete and continuous quantities. In none of the last three meetings were they given compare problems similar to those analyzed here. During the sixth and seventh weeks, pairs of children were interviewed and asked to solve the following compare problems:

(a) Peter has 9 candies. He has 3 candies less than Susan. How many candies does Susan have?
(b) Paul has 9 candies. He has 3 candies more than Karen. How many candies does Karen have?

Eight children solved both problems without difficulties or, after giving a wrong answer, immediately realized their mistake and corrected it. The other four children, who constituted two interview pairs, showed an interesting pattern of interpretation for comparative statements. Detailed analysis of their interviews shows that a correct answer to the main question in a compare problem does not guarantee that the child fully understands the relationships involved in it. It was also found that children may treat a comparison statement as if it were providing information about changes in quantities. Such interpretation will sometimes work in unison with the correct comparative interpretation, thus leading children to a correct answer despite misunderstanding of the problem. Why and how this occurs seems to be, at least in part, a result of their previous school experiences that focused mainly on change problems and keywords.

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COLLEGE STUDENTS' BELIEFS ON NUMBER SYSTEMS:
An Aesthetic Approach to Understanding

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The discipline of mathematics is unique in that it can be interpreted as both a natural science and an art. In the classical sense, mathematics has been known as the queen and servant of the sciences. Although it is perhaps subjective and culturally defined, many have attested to an aesthetic quality of mathematics. Moreover, Aristotle, Poincare and Dirac have all written of its beauty (refd. by Davis and Hersh, 1981). And, I believe few professionals, if any, in the field would dispute that mathematics is at least partly aesthetic in nature. This study explores the aesthetic characteristics of mathematics in relation to the individual learner, and how the perceived aesthetic affects student's mathematics learning.

The study was conducted in a senior-level number systems course offered at Florida State University that is intended primarily for future secondary mathematics teachers. A hermeneutical approach to understanding is employed to uncover what Dilthey referred to as "the intimate connection between experience and expression" (refd. by Ödman and Derdeman, p. 189, 1999). My methodology includes classroom observations, in-depth interviews and various activities to continually encourage the reflective understanding of both the participants' cognition of mathematics and myself as the researcher. Students' drawings of their views of mathematics will be presented with profiles of their aesthetic insights and mathematical comprehensions.

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THE IMPACT OF ATTENDING PROFESSIONAL DEVELOPMENT AS A GROUP OF TEACHERS

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Sixteen primary teachers were involved in a research project (Smith, in preparation) which investigated whether teacher reactions to features considered to be effective for professional development [PD] (Clarke, 1994) could be explained by their beliefs and practice. Initial classroom observations and interviews classified each teacher as either 'mainly instrumental' or 'mainly relational' based on the ideas of Skemp (1976) and Ernest (1989). The teachers, from five different schools, attended a series of workshops and were further interviewed about the impact of the workshops on their classroom practice. One of the features of effective PD was 'Attending as a group of teachers'. This feature was given a high rating by most of the participants although the 'mainly instrumental' teachers as a group placed more importance on it than did the 'mainly relational' teachers. Other related aspects of PD, such as wanting to sit together as a group in 'table discussion', assisted in the explanation of this unexpected result. The impact of this effective PD feature also influenced the extent of trialing of activities in the participant's classrooms. Influences included the way teachers shared with each other when back in the school environment and whether they taught at the same year level. Differences in the extent of classroom trialing of workshop activities could be explained by these influences as well as the relationship that existed between this PD feature and whether the teacher was 'mainly instrumental' or 'mainly relational'.

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TAKING A SECOND LOOK

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Ricardo Nemirovsky
TERC, Cambridge

The Urban Calculus Initiative is a collaborative professional development project involving two public high schools in Boston: the Jeremiah E. Burke and City On A Hill Charter School. A group of ten math teachers and three researchers from TERC meets monthly in an all-day seminar, focusing on the ideas of the mathematics of change. Participants do math together during the seminar mornings, explicitly focusing on their own understandings as learners, and in the afternoons reflect upon and analyze their teaching practices and selected classroom episodes.

We have been interested in the extent to which participant teachers' work in the mornings supports and is relevant to the afternoon work. We have received much anecdotal evidence from the teachers that they value the morning time just to 'do' mathematics. We find that teachers' time and effort concentrating on their own mathematical learning reappears in the form of recognizing similar experiences in what students say and do. The opportunity and structure that the seminar provides in the mornings seems to encourage participant teachers to look more closely at ideas and algorithms they once learned, or thought they knew, often with the results that they understand more mathematics or that previously unsuspected connections emerge. Those same teachers, in the afternoons, have begun to express fresh insights regarding their students' approaches. Where once a teacher might have skimmed over a student's seemingly incorrect answer, or where that teacher might have automatically approved of an idea which seemed 'right,' the participant teachers are reporting that they are asking kids to say more about what they are thinking, that they are asking further questions, or that they are pursuing an idea with a student after class.

This year, all the teachers in the project are writing case studies of episodes in their classes. One teacher describes overhearing two students discuss how the area of a parallelogram changes as one "tilts down" its slanted sides (maintaining the same length for all the sides). The students were arguing whether the area remained the same. The teacher realized that he made a quick decision about who was right but held back from stating his judgment and asked them to have a conversation about this problem after class. He audio-taped the conversation and by working with the tape and the students' work came to see aspects of correct approaches in both students' arguments and to refine his own ideas about the area of a parallelogram.

During the presentation we will examine evidence for the inner relationship between content and pedagogy in professional development (Shulman, 1986, and Warren & Rosebery, 1995). We have found that addressing both at once encourages teachers to think about and understand higher level mathematics while becoming more attentive and sensitive to their student's ideas. Asking teachers to take a second look at the mathematics they once learned has become inextricably linked to having them take a second look at their students' ways of understanding math.

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AN ANALYSIS OF A JAPANESE HIGH SCHOOL STUDENT'S ABILITY TO INTERPRET GRAPHS

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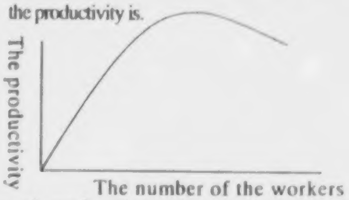
[student A] The larger the number of the workers is, the larger the productivity is. The smaller the number of the workers is , the smaller the productivity is.



I observed a student's interpretation of a graph that shows the connection between workers and production capacity. He saw the labor decrease from the curved line image, but could not read value of the x-axis and y-axis. It's surprising that there are so many students of this type. The question how to teach students how to read the graph is so difficult because we have a few teaching materials. This is a problem because there are a number of such students who can't read the value in the graph even after learning linear function and quadratic function. I conducted an investigation into the ability of Japanese students to read graphs and think graphically.

I think that there are many teaching materials for drawing graphs in Japan but very few are intended for how to read and use graph information. So I conducted a research to find a correlation between understanding how to analyze(interpret) graph information with the skill used in drawing a graph and /or reading graph information. I presented this problem to a 1st grade high school student as follows,(①To read a value from a plotted coordinate ,②To plot a point from a coordinate, ③To express the connection between labor and production capacity (seeing a graph), ④To make a graph of rectangle's area that has the same length, ⑤To express the connection between time and distance seeing graph ⑥To make a table from algebraic expression ⑦To make a table from a point on the graph ⑧To read a maximum or minimum on the graph ⑨To make a graph the connection between a work and a pay. ⑩To make a graph of the connection between time and distance from ground.

As a result of this research, I found out the followings.

1. These ten problems are categorized into 3 groups. A lot of students are poor at the problem belonging to group Ⅲ.
2. There is no interrelation between each group. The gap of student's reaction is widely caused by qualitative difference.
3. I noted an interrelation between group Ⅰ and group Ⅱ. Therefore students can plot a point on the coordinate or read a coordinate, but they can't always read the point on the graph.
4. I noted an interrelation between group Ⅱ and group Ⅲ. Therefore students can read the point on the graph, but they can't analyze(interpret) graph information.

In conclusion, the ability to make use of graph and to process problems mathematically is one of the abilities which are regarded as an aim of our course of study in Japan. Students are poor at the problem belonging to group Ⅲ, which asks them to put the connection they express into algebraic expression and graph. It has been thought that students can get these abilities using the ordinary teaching method. Learning how to draw a graph or reading a graph is not necessarily the way of obtaining the ability to interpret a graph.

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FUNCTION CONCEPTS: WHAT COLLEGE STUDENTS HAVE

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One of the contents of secondary mathematics understood least by students is function in Japan. Thus it was investigated by asking the meaning of some functions in a questionnaire what concepts Japanese college students have for typical continuous functions learned in junior high school and high school.

The responses of subjects were classified into categories such as formula, graph, rule, and functional dependence. The following results were obtained:

1. On linear functions and quadratic functions, right function concepts were acquired comparatively well. There were a few subjects who made no response. It is noteworthy that there were not few subjects (8.7%) who consider the general definition of function to be a property peculiar to linear functions. This response is attributable not only to the fact that linear functions tend to be prototypes (Schwarz & Hershkowitz, 1999), but also to the secondary mathematics curriculum in Japan.
2. On exponential functions, more than half of the subjects did not have the concept of uniform multiple changes of exponential functions.
3. On logarithmic functions, there were many subjects who have only the function image of something using "log".
4. On trigonometric functions, there was no response belonging to the category of graph. No respondent pointed out the periodicity. It may be assumed that uniform periodic changes of trigonometric functions are not fully grasped for most subjects.

THE DIFFICULTIES OF VOCATIONAL SCHOOL STUDENTS IN PROBLEM SOLVING

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Vocational instruction is a subject of increasing interest for mathematics education research (cf: Burton, 1987; Forman & Steen, 1987; Hahn, 1999). In Hungary, a research project was started in 1995 to identify and prevent damage from social and cultural disadvantages in vocational schools. Some initial results are presented in this paper. We had 237 14-15 year-old students in our project. They represented the total number of first-year pupils in five different vocational schools in the same region of Hungary. Five word problems were given to them in October, and no time limits were set. One of the problems was the following:

> We are planning to nail a decorative band onto a corridor wall. For one side of the wall we need one piece, which is 315 cm long, and for the other a 442-cm-long one.
> What length of band do we need exactly?
> How many pieces of band should we buy if it is only available in 2-meter-long pieces?

Some data and comments

i) In many cases the pupils skipped or left out problems.

ii) In hardly any cases did we find explanation of the problem or checking of results. These skills are usually stressed in earlier school years.

iii) In many cases the result merely consisted of numbers, without any kind of word communication, running against the current practice in earlier school years.

iv) The students had many difficulties with the conversion of units, in spite of the fact that conversion techniques ought to have been practised for over 6 years.

v) A high number of pupils built up a solution but failed because they did not check whether their answer was suitable for the given context.

Possible interpretations and further research: One immediate and general interpretation of the available data might be that the routines that were developed in the preceding schooling only confused the students' common sense in problem solving. Other interpretations are related to preceding comments and data:

a) Most of these students had had difficulties in mathematics; the didactical contract with them was probably centred on low-level technical performance.

b) Due to their difficulties in mathematics these students frequently did not reach a solution, so their solution-checking practices were not developed.

c) The prevailing style of classroom work had featured only "ritual" attention to checking, word answering, etc.; the main "value" had been a correct answer.

d) The fear of failing and expectations of poor performance (so frequent in slow mathematics learners) prevented these students from true "reasoning".

Each of these interpretations may offer some keys to interpreting the reasons for some students' failure. In order to find which of these interpretations is the most reliable, further research will be performed, including interviews with the students and analysis of students' exercise books from preceding schooling.

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A STUDY OF THE FEATURES AND TEACHING OF MATHEMATICAL CONCEPTS OF MENTALLY RETARDED CHILDREN

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Generally, physically disabled or mentally retarded children have disadvantages which caused by the handicap. In order to overcome the handicap, a special educational environment must be prepared for the handicapped children. At present, however, we cannot say that any effective and full measures are taken to remedy the situation.

From the viewpoint of arithmetic education, for example, the traditional way of teaching in the arithmetic class does not fit the situation of mentally retarded children (henceforth, MRC). As there is much difference in the ability among the individual MRC, we must prepare an education program appropriate for each of them.

For about 10 years, I have studied mathematics education for handicapped children, especially for MRC. The study has been mainly based on the following two viewpoints;

① To grasp the present situation and problems of education for handicapped children and to search for the most appropriate education for them.

② To understand the characteristics of the cognitive development precisely and to develop the most appropriate ways of teaching.

Here, I will report the results of the study from the viewpoint ②.

In the present study, we set the following research hypothesis:

Research hypothesis: It is possible to make MRC understand arithmetic concepts correctly, if we set arithmetic systems which suit the current state of the children and prepare appropriate teaching materials .

To verify this hypothesis, we have done research for 4 MRC (including 3 Down's Syndrome children and 1 learning-disabled child) and given instructions to them about 25 times a year for 10 years. We have investigated the characteristics of the recognition of number concepts by the Down's Syndrome children and made the education program suitable for them. Especially, as the instruction using computers seemed to be effective, we tried it.

I will introduce the example of the learning method using a computer, which has to be very successful. Concerning the addition of 1 digit numbers which have carrying up, for example $8+7=15$, we analyzed the system of the teaching in detail and programed the system into a computer, and then conducted their learning.

The teaching of the calculation (addition with carrying up) was roughly divided into the following three steps. The first step is the operational activities with teaching tools, the second step is the activities using computers, and the third step is the calculation which is due to numeral expression only.

As a result, one of the MRC (with Down's Syndrome) developed abilitty to smoothly calculate the addition with carrying up. At the presentation of the meeting, the actual scene will be shown with a VTR.

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Daniel: a student with learning difficulties in College but competence in the workplace?

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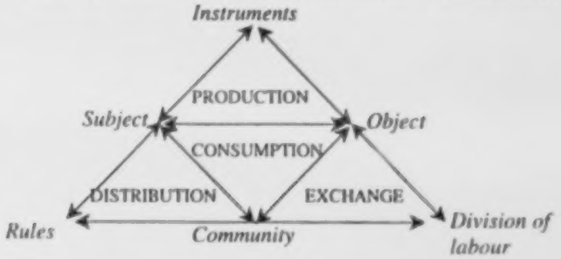
Our project "Using mathematics to understand workplace practice" investigates the ways in which students can understand workplace practices drawing on their College mathematics. We are interested in exploring the well known obstacles to transfer but also the potential to bridge the gap between academic and practical competences. The key question for the project is 'what mathematics is transformable and what is specifically situated?'

We have found the activity systems theory of Engestrom (see, e.g. Engestrom & Cole, 1997) particularly useful in helping to identify features in the systems of College and work which are in conflict and which create obstacles for the student. The schema below illustrates the model. The student on work placement in a sense embodies and lives the contradictions between two activity systems when they move from College to work and try to make sense of the latter with the tools of the former.



This presentation describes one case study in which we explore Daniel in two situations: in College and on work placement at the council gardens department. The contrast between the activity systems of College and workplace seems stark. The College provides a surprisingly threatening social situation, and focuses on strictly mathematical learning objectives which seem to highlight Daniel's weaknesses. He perceives the curriculum as a rush to cover material which he can't easily learn and is frustrated and gives up when he fails. On the other hand, the work system is supportive: the objective of the activity is work-production and the maintenance of quality is paramount. Thus the system encourages teamwork in which the division of labour protects Daniel. But incidentally Daniel is offered many opportunities to learn, and he and his colleagues have a sense that there is plenty of time for him to pick things up. This activity system therefore has a quite different division of labour from the school, as well as objective.

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SEPARATION MODEL BASED ON DÖRFLER'S GENERALIZATION THEORY

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We have already considered the division with both decimals and fractions so far (Yamaguchi & Iwasaki,1999). What we had noticed is that the generalization in the division with fractions is quite different from that with decimals. That is to say, fractions in the former are going up to the unknown number system i.e. rational numbers and their division is universalized as the special case of multiplication. On the other hand, decimals in the latter anchor to the known number system i.e. natural numbers and their division is confirmed as the development of division over natural numbers. These two kinds of generalization are in striking contrast if they have a direction.

Dörfler proposed his generalization model which distinguished between generalizations which were extensional and intensional(1991). We, however, could not explain the above structural difference sufficiently on the single track of his model because it set intensional generalization after extensional one. As a result of this research, we propose the following three points. The first is to propose the alternative framework for generalization, which is called as "Separation Model" based on Dörfler's generalization theory. Our Separation Model distinguishes the generalization process of division with fractions from that with decimals by calling the former extensional generalization and the latter intensional one. The second is to clarify the cause of much difference on performance between both making an expression of division with fractions and that with decimals as one example of application for our model. The third is to improve teaching and learning process of division with fractions changing the traditional perspective to the new one on the basis of the Separation Model. Our approach is summarized as the following Table 1. The teaching experiment and its analysis showed the relevance of the new teaching for the cornerstone of algebraic sense.

Table 1. The Comparison of the present framework with the alternative one in the case of division with fractions

	Present framework	Alternative framework
Making an expression	Schema of proportion	Schema of comparison
Understanding the algorithm	Schema of proportion	Deduction from properties of fractions and rules about division

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ANALYZING CHILDREN'S PROBLEM-POSING RESPONSES

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Problem-solving and problem-posing have both been identified to be at the heart of doing mathematics. While there has been much research into mathematical problem-solving, problem-posing research is only gaining momentum in recent years.

This paper reports one aspect of an on-going investigation into children's mathematical problem-posing. Such an investigation requires the development of a framework to analyze responses to problem-posing tasks. This paper aims to describe the development and validation of such a framework and to use the framework to investigate the relationship between problem-posing and problem-solving.

Various frameworks have been proposed to analyze responses to problem-posing tasks to reveal the complexity of the responses. Silver and Cai (1996) used the number of situations in the posed problems as a measure of mathematical complexity. Marshall (1995) has earlier identified five basic situations that are found in arithmetic word problems. Word problems with varying number of situations and categories of situations were written, controlling other factors. Fifteen word problems were randomly arranged and given to 250 subjects. The results indicate that the number of categories in a word problem as well as the number of unknown information that has to be handled simultaneously make the problem more complex, and hence more difficult. The empirical findings refine the framework proposed by Silver and Cai (1996).

The framework was subsequently employed to analyze written responses by 240 children in grades three and five to five problem-posing tasks. These children also completed a six-item problem-solving test. The problem-posing ability and problem-solving ability of these children were quantitatively compared. The findings suggest that although problem-solving ability and problem-posing ability are related, they do not involve the same processes.

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Informal knowledge in children before learning ratio in schools formally

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Previous studies pointed out that children have a rich informal knowledge in multiplicative one as well as additive structure. For example, some investigators showed such rich informal knowledge in fraction, decimal fraction. However, almost all of previous studies paied very few attention to ratio. It is highly hard to understand ratio concept in middle school students as well as elemenatary school ones in Japan. So, it is also very hard for teachers to teach ratio concept. Recent investigation indicated that students before learning ratio in their schools acquired informal knowledge on ratio (Yoshida & Kawano, 1999). Many of students understood part-whole relation in ratio, composition or decomposition in ratio.

The present study aimed to investigate how students before learning ratio in school formally solved ratio problems by using informal knowledge. The subjects were thirty-five fifith graders who did not learn ratio concepts in their schools and thirty-seven sixth graders who had learned the concepts for ten months before. One of problems used were [There were 40 beans. What is 50 % of the beans?] Format of other problems were just same to it except percentages. Percentages used in other problems were 25%, 75%, and 90%, respectively. Problems was presented in 50, 25, 75, and 90%. Order of presentation was fixed. Each problem was given to each student. After he/she read the problem, the experimenter instructed to solve it by pencil and paper or by using real beans on right corner of a desk. In a case of sixth graders, if students solved all problems by computation, they were required to solve them without relying upon computation. Or vise versa.

Correct percentages in 50, 25, 75% problems for fifth graders were 67, 52, and 46%. On the problem 90%, there were many answer of "35". Because it is very difficult for them to solve the 90% problem exactly, we set two criteria of both strict (36) and lenient (35) answers. Correct percentages in strict and lenient were 9 and 40%. None of fifth graders learn ratio concepts, neverthless, 40% of them were able to solve the 90% problem which was fairly hard for students, based on the lenient criteria.

How did they solve such difficult problem? In order to answer this question, we have to analyze strategies in which students adopted in solving the problems. We will show such strategies in poster session.

A STUDY OF EVERYDAY CONCEPTS AND MATHEMATICAL CONCEPTS BASED ON VYGOTSKY'S THEORY

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Vygotsky's theory seems not to be examined enough in mathematics education. Despite some Japanese mathematics teachers recognize the importance of his theory, in particular the idea of *Zone of Proximal Development* [ZPD], implications for the teachers' roles in the process of teaching and learning mathematics concepts have not been clarified. It is needed to materialize Vygotsky's theory to be useful in the practice of mathematics education. The purpose of this paper is to examine the relationship between *everyday concepts* and *mathematical concepts* based on Vygotsky's theory, and to determine children's everyday concepts in fraction.

Vygotsky (1975) mentions *everyday concepts* and *scientific concepts* in the manner of related to ZPD. In the present paper, the term of *mathematical concepts* is employed for scientific concepts. Everyday concepts depends on concrete contexts, whereas mathematical concepts refers to a general system. However, these two concepts complement each other when children acquire meaningful and mathematical concepts. Consequently it is important to make clear the relationship between them. Such relationship will be understood much better when adopting the idea of *sublating*.

In this paper the word of *sublating* means a sequence of three activities: canceling, lifting and storing, related to everyday concepts and mathematical concepts. When children meet certain mathematical concept and are aware of an inconsistency with a part of their everyday concepts, which are opposite to the mathematical concept, children cancel the inconsistency and lift both the mathematical and everyday concepts to higher levels, and they store these two concepts as a whole, which is so-called *sublated concept*, that has both a concrete context in their real life and a general system in mathematics.

An interview research, whose purpose is to find children's everyday concepts of fraction, has also been conducted. Subjects consisted of six Japanese third graders. They have never studied fraction in school, but some of them seemed to have been studied in cram schools. As a result, some cases of children's everyday concepts in fraction are determined. For example, even though some children were aware of the meaning of "one third" in different size of circles, they tended to pay attention to the differences of the quantity. Also, some children had both sublated and everyday concepts in fraction. For example, one child knew that "one second" means half, but she could not regard "one second" as half of one thing but one part of two things.

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POSTER PRESENTATIONS

RELATIONSHIP BETWEEN PROBABILITY AND OTHER LANGUAGES USED IN THE SCIENCES: THE PYTHAGOREAN "ARITMO-GEOMETRY" EXPANDED TO D-DIMENSIONS AND IN THE CONTINUUM AND THE DISCRETE SPACES. A DIDACTIC PROPOSAL

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Starting from:

1) experimentation with students, ages 10-24, and teachers
2) the responses of students and teachers to several questionnaires
3) an analysis of the peculiarity of probability and its teaching
4) a short analysis of the new requirements of our society
5) some of Bruno de Finetti's suggestions.

I propose for consideration to establish a close relationship between probability and other languages, particularly, geometry, used in the sciences. Combinatorial calculus, physics, analytic calculus, numerical analysis, and theory of numbers are also related to geometry and probability.

My purposes are to promote:

a) a better understanding and mastering of probability, "seeing probabilistic measures" and their properties in geometry and in some other languages used in sciences
b) a better consideration of different cognitive styles
c) a growth, in the students, of a positive correlation between inductive and deductive thinking, developing the abilities to visualise and to generalise.

More precisely the above will be reached using the Pythagorean "Aritmo-geometry" expanded to d-dimensions (dinamically, with Cabri also in 4-dimensions) and in the continuum and the discrete space, considered in close connection.

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THE FROG AND THE PRINCE CHARMING:
CHANGES ON PUPILS' SOCIAL REPRESENTATION ABOUT MATHS

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Mathematics is a very important subject for pupils' vocational and professional choices. At the same time, in Portugal, it is also the subject which has the highest rate of underachievement and the one pupils rejected most. Pupils who fail describe it as difficult and that can only be learnt by those who have very high abilities.

The importance of pupils' beliefs about Mathematics has been emphasized in many studies (Brown, 1995; Civil, 1995; Rodd, 1997). A previous study (César, 1995) showed that 7th graders from a school in Lisbon had quite traditional ideas about Mathematics. They thought Maths was important for their future life but they associated it mainly with computation and "memorizing things". Another study (César, 1996) stressed the influence of innovative practices on pupils' ideas about Maths and on the changes that took place during a school year's work.

Following these previous studies we started a project called *Interaction and Knowledge* which aim is to implement peer interactions in Maths classes as a way to promote pupils' attitudes towards Maths, their socio-cognitive development and their school achievement. The data of this poster are focused upon students' social representations about Maths, comparing their ideas in the beginning and at the end of the school year. Our sample comprises 7th to 10th graders attending 20 classes from 4 schools, both urban and rural. Our analysis is based upon their phrases and drawings about Maths in an attempt to use instruments of a more projective nature.

In the beginning of the school year many students had a negative image of Mathematics, as we can see in their phrases and drawings. There were also great discrepancies: a small part of them love it; all the others hate it. At the end of the school year and after being part of daily practices that are based on collaborative work pupils' ideas about Maths had deeply changed. Working in dyads led them to a more positive attitude towards Maths and so they slowly transformed the frog into a prince charming.

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NEW EXPERIMENTAL COURSE OF GEOMETRY

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Today's development of school education is characterized not only by the creation of new curricula and adoption of new subjects, but also by changing points of view on the structure and aims of school education in general.

We will give a short description of our general points of view on school geometrical education and explain main characteristics of the new "Geometry 6-9" (for pupils aged 12 to 15 years) course we are developing.

Today the primary school, the contents and methods of primary mathematics are rapidly developing in Russia. It is clear that primary school can and must promote the formation of the geometrical culture of pupils and that some minimal amount of notions should be precisely defined.

The most important stage of the mathematical education is lower secondary (grades 6-9). Our course "Geometry 6-9" is based on geometric notions learned by pupils in primary school. Primary school must provide basic knowledge and skills demanded by official standards of mathematical education for secondary school, reveal pupils' abilities and help them to determine their further ways of education and professional activity.

In grades 10-11 geometrical knowledge should be used by pupils with consideration of their interests, abilities and the profile of their future educational or professional activities.

The main strategy of learning the course "Geometry 6-9" is "I am in the (3-dimensional) space". Traditionally, in Russian secondary school geometry course consists of two main parts: planimetry – geometry on the plane, and stereometry – geometry in the space. However, geometry as a science arose from the experience of the mankind, and in the nature there are no purely flat objects. Further shortcomings of the traditional approach are the following:

- studying planimetry and stereometry separately, one encounters a lot of duplications;
- separate study of properties of figures on the plane and in the space does not allow a pupil to see many common results in geometry, and a pupil regards plane geometry and space geometry as two different sciences;
- applications of planimetry are artificial and oversimplified, they do not reflect the connection of geometry to with the reality adequately; however, pupils rather early study a lot of things about the 3-dimensional nature in courses of physics, chemistry etc.
- pupils leaving school after the 9-th grade do not have the opportunity to study space geometry.

21 PUZZLE BLOCKS – A TANGIBLE GAME INTEGRATED WITH A COMPUTER PROGRAM

Bat-Sheva Ilany, Haim Orbach

Beit Berl College & Or International Co.

The program combines tangible aids and games with activities performed on the computer. In this manner, the child is engaged on two levels: motor activities carried out with tangible aids and games and the abstraction step, performed on the computer. The computer program enables each child to progress at his own development rate and ability level. Part of the software is open, enabling the child to create activities on the computer, then apply these to working with tangible aids, and vice versa. The construction created by the child, tangible and on the computer, can be printed out and used as activity cards for him, and for other children. The child thus experiences a two-way transition from a three-dimensional space – the cubes, to two-dimensional space – the computer.

The game contains 21 building blocks designed on special principle and cards for activities.The game enables the preschool child to learn to differentiate shapes, colors, sizes and to adapt a tangible shape to a drawing of that shape. The game also facilitates learning through the discovery that blocks of varying shapes can be the same size.

The computer program is designed on the principle of the game, facilitating a greater variety of activities, and their deliberate grading. The grading is expressed in the number and size of the blocks and the position of blocks that are to be used for building on the card. Namely, at the first stage the blocks are presented in a manner that the child only needs to move and rotate them, while at the more advanced stages, the blocks are asymmetrical, requiring them to be turned over as well. Each child is offered five different possibilities of working: The building card specifies the block colors, Only uncolored blocks appear on the card, The building card is inlaid, Only a contour of the shape is provided, Free form creation.

WHAT IS THE DIVERGENT THINKING? : FOUR TYPES OF LOCAL DIVERGENT THINKING

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One of the most popular cognitive theories of creativity is Guilford's theory of divergent production, and creativity has come to mean divergent thinking in much research in, assessment of, and theorizing about creativity (Baer, 1993). "The unique feature of divergent production is that a *variety* of responses is produced" (Guilford, 1959, p.473). However, if divergent thinking were considered as global thinking that produces a variety of *solutions* or *answers* to a given problem in school mathematics, the concept of divergent thinking would become narrow. Because it is not to say that divergent thinking does not come into play in the total process of reaching a unique conclusion (Guilford, 1959). That is, divergent thinking has a necessity to be considered as a local thinking that emerges throughout global process both of divergent and convergent thinking.

Another issue suggested by Baer (1993) is that "divergent thinking may play an important role in creative performance if one knows when to use it" (p.69). Therefore, it is necessary for mathematics teachers who attempt to foster children's creativity and creative thinking, to grasp the essential concept of divergent thinking and how and when it comes into play in the process of creative thinking.

The purpose of this paper is to identify and classify local divergent thinking, which could be considered to contribute to the number and variety of children's responses, in order to foster children's creativity and creative thinking effectively from the viewpoint of mathematics education.

Four types of local divergent thinking that can be considered as different from each other are identified in this paper. They are the following.

****Divergent Perception***: This type is the thinking activity for perceiving diverse attributes in the object at hand.
****Divergent Remembrance***: This type is the thinking activity for accessing to diverse knowledge by using the perceived attributes and accessed knowledge as a clue.
****Divergent Transformation***: This type is the thinking activity for transforming the perceived attributes and accessed knowledge to diverse information.
****Divergent Connection***: This type is the thinking activity for connecting perceived attributes and accessed knowledge to themselves in diverse ways.

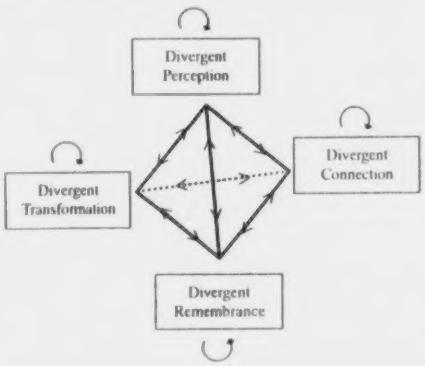

Fig. 1: The Structure of Interrelationship between Four Types of Local Divergent Thinking

These types of local divergent thinking do not emerge independently throughout the total process of creative thinking, because of their complex structure of interrelationship (see Figure 1).

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FEATURES OF CHILDREN'S SPATIAL THINKING AT VAN HIELE'S LEVELS 1 AND 2

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In psychology and mathematics education, various problems about spatial ability, visualization and the relationship between mathematical performances and cognitive actions of students have been studied. In this study, I define *spatial thinking* as the integration of spatial ability and mathematical thinking, and characterize it globally from the points of both van Hiele's levels of thinking and its components ("imaging and visualization", "representation" and "spatial reasoning"). In this presentation, I especially focus on and describe the modes of thinking at levels 1 and 2 in respect of *image schemata* (Dörfler, 1991).

At van Hiele's levels 1 and 2, it is the feature that is transition of object and way of thinking from material to its properties through abstracted shape. Especially focusing on verbal representation in the components of spatial thinking, mathematical languages aren't used appropriately by children nor significant for them. That is because for example, a way of recognition at such levels is based on a naive sense (regular, beautiful,...) and the function of object (arranged in order, round,...), hence there is a difference of constructing image schemata between natural language in everyday context and mathematical one in classroom context.

Besides, since children at so-called low level cannot manipulate visual images freely enough, image schemata they construct aren't 'rich'. That is to say, for children don't recognize relationships among components of object, it is great stride to do cognitive actions on the geometric-figurative schemata, such as interpretation, application, projection and transformation (Dörfler, 1991).

As a consequence, I could list following features of children's spatial thinking at van Hiele's levels 1 and 2:

- Children construct the single figurative image schemata under the effect of their naive sense (for example, when asked to draw a triangle, they only do regular or isosceles one.)
- Children construct the operative image schemata based on everyday life under the effect of the function of the object (for example, they don't recognize a 'pointed' triangle as a triangle, rather a pointed material.)

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Mathematics Education in Germany and in Japan

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Yonago National College of Technology
Shimon Uehara
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In German upper secondary schools Mathematics is taught in various applied ways. For example teachers make examples such as to buy a car with many options. They calculate how to buy a car in an optimal way i.e. to pay how much money in a month. Their way of teaching Mathematics is not only very practical but also very theoretical.

On the other hand in Japanese way we pay too much heed to calculate the expressions (i.e. formulae). For example in differential and calculus Japanese students can calculate very well and very fast. But the big problem is that they don't know what they are doing.

It is said that German people and Japanese people have many things in common in their characters. But in my opinion the differences on Mathematics Education between two countries are very big. We should like to talk about these differences in a form of poster session in PME24. Hiroshima.

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EFFECTS OF TEACHING FOR DEVELOPMENT OF METACOGNITIVE ABILITY

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SUMMARY

The purpose of this paper is to investigate the effects of teaching for developing of metacognitive ability. This experiment involved 27 fourth grade children. According to the purpose, group teaching sessions were carried out for an experimental group that consisted of 4 pairs of children. In the group teaching sessions a researcher promoted a pair of children to do metacognitive activities through the framework of teaching for development of metacognitive ability (Table 1). These sessions were not carried out for a control group that consisted of 19 children. Instead, a pretest and posttest were carried out for all 27 children.

The main findings of this investigation are the followings :

- Both metacognitive and cognitive growth of the experimental group are higher than those of the control group
- In the posttest, some children in the experimental group did some metacognitive activities that they had not done in the pretest

METHOD

Group Teaching Session Table 1 is the framework of the teaching. It is based on Schoenfeld (1987) and others.

Pretest and Posttest In the tests, each child was asked to solve the problem on the work-sheet and to answer the stimulated recall questionnaire. The work-sheet was used to analyze his/her problem solving process, and the stimulated recall questionnaire was used to represent his/her metacognitive activities.

Analysis These problems of the pretest and posttest were the same. Then from the tests, children's metacognitive growths and cognitive growths were respectively identified as followings.

Table 1 Framework of teaching for development of metacognitive ability

Ⅰ. Solving the problem by oneself.
Ⅱ. Talking about cognitive activities.
Ⅲ. Talking about metacognitive activities.
Ⅳ. Using the metacognitive activities for a similar problem.

Table 2 Framework of investigation of metacognitive activity

1. Method of Investigation
   (1-1) Solving a problem on the work-sheet
   (1-2) Answering the stimulated recall questionnaire
2. Method of Analysis
   (2-1) Scoring the mathematical problem solving processes
   (2-2) Counting of metacognitive activities

Metacognitive growth is defined as

[the number of his/her metacognitive activities on the posttest]

—* [the number of his/her metacognitive activities on the pretest].

Cognitive growth is defined as

[the marks at his/her work-sheet on the posttest] — [the marks at his/her work-sheet on the pretest].

The results of teaching will be reported in this poster and the processes of teaching will be presented on video.

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* This means to subtract.

A STUDY OF THE PROBLEM OF TEACHING MATERIALS UTILIZING A GRAPHICAL ELECTRONIC CALCULATOR

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SUMMARY

In this paper, we discuss about a small computer, especially, a graphical electronic calculator. We consider about its present situation and its problem.

§ 1 INTRODUCTION

In the second half of the 20th century, the computer technology was innovative. A graphical electronic calculator was developed in 1985. T-3 which is called T cubed-Teacher Teaching with Technology begins to attract attention and then a graphical electronic calculator has begun to be used in case of mathematics education.

Therefore, the purpose of this paper is to consider the problem that it should attend to by the use of a graphical electronic calculator.

§ 2 THE PURPOSE OF A GRAPHICAL ELECTRONIC CALCULATOR

There are two purposes to use a graphical electronic calculator for the use of a computer in case of mathematics education.

Firstly, it is important to how to make lead to the conclusion having to do with mathematics more than the way of the use of computer. Secondly, the real-model is complicated to compute more than the ideal-model. However, it is to have the characteristic of which it is possible to let out a result having to do with mathematics by finding out it without the computation by hands.

§ 3 THE PROBLEM WHICH IT SHOULD ATTEND TO THE USE OF A GRAPHICAL ELECTRONIC CALCULATOR

There are three problems to use a graphical electronic calculator.The first problem is how mathematical model use. Because the real-model has some various elements. The second problem is the range of the graphical scene. Because students has to know how mathematical function. The third problem reduced scale on the plane of the graphical scene.

§ 4 THE INVESTIGATION BY HIGH SCHOOL STUDENTS AND CONCLUSION

In 1999, these problems was investigated by twelfth grade students.

As the conclusion, it found that the knowledge with some degree having to do with mathematics was to be necessary for the use of a graphical electronic calculator.

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A REPRESENTATIONAL MODEL ON THE CONCEPT OF GEOMETRY

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There have been many researchers who investigated into the concept of geometry. I would like to focus attention on the representations of the concept of geometry. The main purpose of this research is to reflect on the meanings of the representations on the concept of geometry and clarify the epistemological model. For this purpose I considered two issues as follows and insisted on the representational model.

1. What are the characteristics of representations used in the teaching of geometry?
2. What is the epistemological model on the concept of geometry?

The representational model is an epistemological model on the concept of geometry that is constructed with two types of concept: mathematical (objective) concept and individual (subjective) concept. These two concepts are characterized by difference between output and input of information in the teaching of geometry or between external (physical) representation and internal (mental) representation.

The mathematical concept of geometry holds epistemological meanings in the external representations that are informed by teachers in the teaching of geometry. There are five styles of representations on the mathematical concept of geometry: realistic, operational, figural, linguistic, and symbolic representation. And we have to clarify the crucial roles of them in the teaching of geometry. For example linguistic representation is a representation that is used natural language. One of the representational style is "term", that is to say the names of the concept of geometry and another style is "sentence": definitions, characteristics, and propositions. Figural representation is to represent shapes of geometrical figures and relationships among figures. And the epistemological meanings of figural representation are summarized into five points: spatiality, visuality, entirety, typicality, and generality.

The individual concept of geometry is the information that is stored in student's memory subjectively and represented internally by two types of representation: verbal representation and imaginary representation. Names and definitions of geometrical figures and sentences are all represented by verbal representation. On the other hand shapes, distances, positions, and directions of geometrical figures can be represented by imaginary representation.

This idea of the individual concept of geometry is similar to the "dual-code theory" in cognitive psychology, the "concept definition and concept image" in Tall & Vinner's theory, and the "figural concept" in Fischbein's theory. But the fundamental difference is that the individual concept of geometry in my theory is represented both by verbal representation and by imaginary representation simultaneously.

For example some students can recall names and definitions of both parallelogram and rhombus by verbal representation, nevertheless they can't identify these shapes by imaginary representation if geometrical figures are placed in the unstable positions.

THE CHANGING MATHEMATICS CURRICULUM IN SOUTH AFRICA. -JUST ANOTHER GRAND NARRATIVE? OR PURE SIMULATION?

Agatha Lebethe, School Development Unit, University of Cape Town, South Africa.
Gabeba Agherdien, School Development Unit, University of Cape Town, South Africa.

The poster describes the results of a research conducted among teachers described as 'key' teachers by the School Development Unit (SDU) at the University of Cape Town. These teachers have developed a strong relationship with the SDU, have been trained to undertake a range of development activities, participated as consultants and workshop and course presenters. These teachers have also delivered at regional and national mathematics education and inservice conferences. The study investigates these teachers conception of the specific outcomes and the application the mathematics classroom considering their strong relationship with a mathematics inservice project. The objective of the study is to contribute towards teacher inservice, to conversations around the effect of the implementation of such a complex system such as OBE on teachers who received minimum formal preparation and training and very little change to the material resource base to enable the new curriculum. We see the research as ongoing and necessary to demonstrate the distance between policy and practice, between the intentions of Government and what teachers experience and so contribute to the question posed by a noted South African Academic, Jonathan D Jansen, 'What does the South African experience tell about the classroom practice?"

The poster will be presented in a postmodern form and use symbols and images to question whether the implementation of a new curriculum is not another Grand Narrative or have we entered a period in South African Education that can be referred to as an age of simulation, in which advanced forms of fakery and illusion are now dominant elements of culture and society.

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Moments of Decision Making in Teaching for Understanding

Judith A. Mousley
Deakin University

"The understanding" has been a focus of the philosophy of education from at least the time of Aristotle. Teaching for understanding was a recurring theme of early debates on learning, and this stream of thought can be followed through to today's emphasis on, for example constructivism, socio-cognitive theory, cognitively-guided instruction, and connected knowing—among other relevant theories. Much of PME's work has centred itself around students' understanding of particular concepts.

There is an implicit expectation that children will construct their own understanding of the mathematics they use, and at times this is made explicit. For instance, the influential and overarching curriculum document in Australia, presents a set of key learning principles, the first of which points out that learners construct their own meanings from, and for, the ideas, objects and events which they experience. Similarly the USA's *Curriculum Evaluation Standards for School Mathematics* and the *Professional Standards for Teaching Mathematics* also stress the importance of children making their own mathematical meanings—although these documents take a more socially-situated view of learning than their Australian counterparts.

Despite such indications, and the fact that teachers and teacher educators frequently expound the importance of teaching mathematics in ways that allow children to make meaning, there are tensions between statements of expectations and features of the learning and teaching environment. This poster explores some of those tensions, and the ways that they are played out at particular "moments of decision making". These moments include teacher's readings of curriculum documents, their conversion of given learning objectives into pedagogical activities, the introduction of key concepts through these activities, teacher-student interactions, and teachers' assessing what was learned.

In the project reported, case studies of four primary-school teachers were undertaken. Each teacher was videotaped for about ten lessons, and pre- and post-lesson interviews were held. One longer interview with each teacher, the curriculum documents they used, and a questionnaire provided further data on the teachers' constructions f their own roles in mathematics lessons.

The poster presents findings that relate to the tensions between (a) the expectation that children will understand the mathematics they are expected to use, and (b) the realities of teacher's work as it is situated in institutions. Each of the moments above creates its own set of potential roles. The poster presents what one teacher was thinking at these moments. The teacher's decision-making processes were explored through video-stimulated recall and interview.

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A STUDY OF DEMOCRATIC COMPETENCE THROUGH MATHEMATICS EDUCATION

Recommendation for the implantation of critical mathematics education in Japan

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Shiga Oral School for the Deaf

The purpose of this study is to suggest that democratic competence through mathematics education, modeled the schools of critical education in England and Denmark, be promoted in Japan from now on. It, as defined by Skovsmose, can not be defined only by reference to technical competence at creating and using models or technologies (mathematical application), it must also involve the reflective knowledge necessary to reconstruct implicit mathematics and explicate the interests and intentions which brought them into existence (reflection through mathematics).

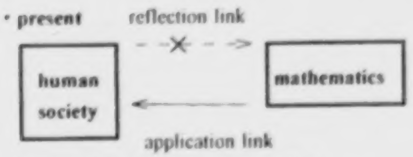

Fig. 1: Mutual relation to two links

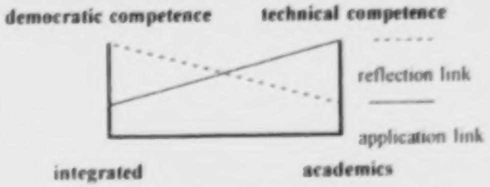

Fig. 2: Integrated learning and two links

The following information in Keitel's paper: "a re-analysis of the political and scientific debate of environmental problems", is seen to be of use in building up our practice in Japan.

《Example》 In those projects (e.g. an analysis of the contamination of water by local industrial production in a North Italian community (OECD-CERI, 1991), or a re-analysis of the political and scientific debate on the building of a bridge over the Northern Belt in Denmark (Christiansen,1994), pupils are confronted with the practical social use of mathematical models and political argumentations underpinned by mathematical methods of manipulating measured data. They are also designing mathematical models to cope with contamination issues or the economic arguments used to justify decisions by those who advocate the building of the bridge. They are going further to inform the public of their findings, presenting these at public hearings, and pressing for alternative courses of action. Here mathematics is studied and used within a highly political context existing in reality and actually influencing daily life of pupils and their environment.

Japan's mathematics education should lead to both mathematical application and reflection through mathematics. In mathematics education it was overlooked to begin to ask social questions such as "For whom and whose benefits?" Planning to introduce the new subject "Basic Mathematics" and "Integrated lessens", we will need to incorporate students to confront with reporting an social investigation such as Project methods devoting to reflection through mathematics especially.

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THE SPECIFIC TECHNIQUE OF CUSTOM REQUIRED TO NEGOTIATE MATHEMATICAL MEANING

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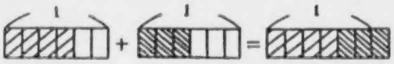
L.Wittgenstein pointed out that the idea "the rule is understood by the interpretation" is fallacy. And he insist that we should admit the existence of the grasp of the rule without interpretation. The reason why we can do some acts by following the rule is not because the game consist of some specified rules, but just because we do the practice customary based on no ground, that is without conceiving other possibilities. It is not the agreement of the interpretation, but the agreement of the judgement that is most essential to play the language game and to talk with each other.

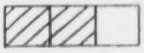
From this point of view, the customary practices based on no ground in mathematics learning is essential to negotiation of mathematical meaning.

For example, when we negotiate the meaning of fractional addition, someone might show figure 1 to explain the calculation $2/3+1/2=4/6+3/6=(4+3)/6=7/6$. Usually we believe that one can perceive mathematical truth from figure 1 directly. And we believe figure 1 show the sum of two quantity by connecting two figures based on the transcendental recognition that congruent figures are equivalent.

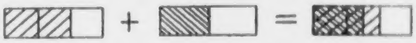


(figure 1)

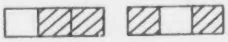
But we require using some specific techniques as the assumption to perceive $2/3+1/2=7/6$ from figure 1. For instance, we use the right figure to shown 2/3 . And we use figure 1 to express the operation of addition.



It is the specific technique to do so. If we use the other additional technique by which we unite two figure as follows, we cannot explain the fractional addition with figures.



The techniques are customary because we use them with no reflection on the reason why we should do so, such as the reason why we draw the rectangle, the reason why we draw the slash on a part of the rectangle, the reason why we vertically draw the delimitation line and the reason why the capitation of the quantity can be shown by dividing length equally. As the evidence, according to a certain elementary school teacher, not a few children do not understand figures below shows 2/3.



It is most important educational task for teachers to form some customary practices or techniques that are required to negotiate mathematical meaning.

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HIGH CONTEXT AND HIDDEN AGENCY IN JAPANESE MATHEMATICAL DISCOURSE: A VYGOTSKIAN PERSPECTIVE

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Recent research has a common and persuasive vision of mathematics classroom as socioculturally mediated milieu. Different classroom cultures mediate different values with respect to classroom interaction, especially to mathematical discourse.

As Leont'ev (1959: 513) illustrates, in classroom discourse, teacher assume an authoritative position and ask *instructional question* (the question to which teacher knows the answer), which consists of three-part exchange: Teacher initiation-Student response-Teacher evaluation. Such specifically organized interactional patterns in the classroom may influence the development child's mental functioning. Newman and others (1989) provide a detailed description of functions of the last turn. It functions repair system and acts as "a gatekeeper" which let the correct responses into the lesson and keep out negative answer, and that such discourse type gives high-achieving students extra support and get them richer. Ohtani (1996) illustrate that, in mathematics classroom discourse, teacher's telling mathematical definitions and conditions serve such a gate keeping role in which decontextualized representation privilege over contextualized vernacular representation. Telling definition and condition obtain its privileged status because they are characterized by high context and hidden agency.

Such tendency as high context and hidden agency in classroom mathematical discourse may be reexamined in light of Japanese culture. Japanese value implicit communication that requires speaker and listener to supply context without explicit utterances and cues. This tendency is typically found in leaving the sentences unfinished. In Japanese discourse, it is often that agency or action are hidden and left ambiguous. Many English verbs are originally transitive or *do-verb* and intransitive verbs become passive if an agent is mentioned or implied (Alfonso & Nishihara, 1989). For example, in mathematics classroom, when we solved the problem, we would say "*Toketa*" rather than "*Toita*". In English, when we introduce definition, they would express in do-verb: "We define". In Japanese mathematics classroom, teacher often introduces definition in intransitive sense ("*Sou Natte Iru*") as if it is beyond ones concern.

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*This research has been funded by the Ministry of Education Japan.

PUPILS' VIEWS OF MATHEMATICS TEACHING IN TATARSTAN

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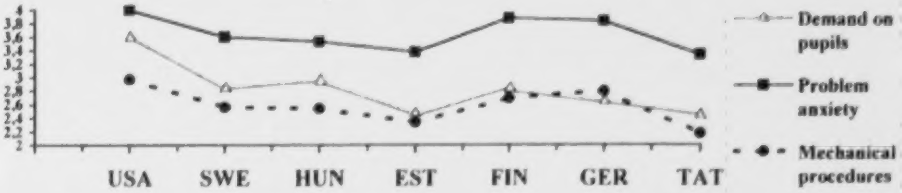
This is a comparative study of pupils' views on mathematics teaching in Tatarstan (Russia) and 6 countries: USA, Sweden, Hungary, Estonia, Finland and Germany. Data were taken from recently published papers (Graumann&Pehkonen, 1993; Pehkonen, 1994; Pehkonen&Safuanov, 1996). All the data were gathered with the help of a questionnaire consisting of 32 structured statements about mathematics teaching and learning for which the pupils were asked to rate their views on a 5-step scale (from 1 = completely agree to 5 = completely disagree). The sample for each country consisted of more than 200 pupils. Using statistical and heuristic methods, we have revealed 6 (not disjoint) clusters and checked means of responses for each country across these clusters. We obtained the following mean values of responses for countries and clusters:

Clusters \Countries	USA	SWE	HUN	EST	FIN	GER	TAT
Demand on pupils	3,6	2,85	2,95	2,45	2,85	2,65	2,45
Democracy	2,18	2,40	2,38	2,20	2,31	2,36	2,54
Independent work	3,10	3,07	2,93	3,15	2,88	3,18	2,83
Problem anxiety	4	3,6	3,53	3,37	3,87	3,83	3,33
Mechanical procedures	2,97	2,56	2,54	2,34	2,69	2,79	2,17
Problem orientation	2,7	2,7	2,76	2,56	2,42	2,6	2,4

Note: the lower numbers correspond to higher agreement.



We see that for clusters "Demands on pupils", "Problem anxiety" and "Mechanical procedures" pupils in Tatarstan and Estonia (i.e., representatives of regions of the former USSR) displayed highest agreement. Indeed, strong demands on pupils, learning mechanical procedures and quick solving of large amounts of problems were cultivated in Soviet Union.

Generally, one may conjecture that in forming pupils' views of mathematics teaching/learning not only social circumstances and educational policy play important role, but also the cultural and historical traditions.

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THE DEVELOPMENT OF THE SUPPORT SYSTEM FOR ACTIVATING STRUCTURAL THINKING

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Seibu Junior High School

We propose the support system for activating structural thinking by utilizing computers in this paper. This system makes it possible to judge the level of students' structural comprehension of their lessons.

The outline of this system is as follows;

- The teacher selects important ten to twenty learning elements from a unit of the textbook and gives the cards of learning elements to the students.
- The students input the structural relation among the learning elements to the computers.
- The students repeat the procedure from twice to five times before they complete a concept-map of the unit.

The computers provide feedback to the students, depending on the students' levels of comprehension. This feedback is broken down into 13 categories.

In general, the fewer the number of attempts, the less precise the information provided for students becomes. By increasing the number of attempts, students can receive more detailed feedback. The students were instructed to use this system in their mathematics classes. The students' performance and the effectiveness of the system were studied.

Fifty students in the third year of the junior high school participated in the study. The results of the study show that the mean values of the transfer coefficient increased exponentially in proportion to the number of inputs. This is because the feedback provided by the computers to the students was appropriate. This shows that the developed system was effective.

Furthermore, with cluster analysis, two different patterns were seen in the increases in the mean values of the transfer coefficients in proportion to the number of inputs. One of the patterns represents the exponential increase while the other illustrates the logarithmic increase.

Concept mapping and writing: Implications for continuous teacher education

Vânia Maria Santos-Wagner, Universidade Federal do Rio de Janeiro, Brazil
Valéria de Carvalho, Universidade Paulista, Brazil

Concept map and writing have been used as ways to help students learn and externalize their thinking about mathematics (Santos & Kroll, 1992). In this study we used concept mapping and writing as tools to enhance teachers' communication and to provide them a way to reflect and develop awareness of what they think about teaching. The main goals of the present work were: a) to propose a form of on-site inservice education through the collaborative elaboration of a pedagogical approach, and b) to explore the role of reflection in the processes of teacher enhancement and professional knowledge acquisition. This was an interpretive investigation in which the close collaboration of one of the authors, who was also a mathematics teacher in the school, with two other teachers provide us the scenery of an action research project. Data for the study included: semi-structured interviews; concept map construction and explanatory written texts; joint meeting for course and lesson planning; informal conversation; exhibition of videos followed by joint discussion, problem solving activities and reflections. In sum, the researchers wanted to investigate the potential of collaborative work and teachers' reflections to the enhancement of teaching practice and to the development of teachers' metacognitive awareness (Olson, 1997; Santos & Nasser, 1995). The power of concept mapping and writing come into place exactly when the authors pursued the aforementioned goals and implemented the work with the teachers. In this presentation it will be showed the concept maps and the texts produced by two teachers who participated in the study and the evolution process of their reflections and metacognitive awareness. It will also be discussed the implications of this form of collaborative inservice education mediated through the use of concept mapping and writing as tools to enhance awareness. This report is part of a research project undertaken in a school environment to investigate: What contributions to the professional enhancement of mathematics teachers could have the collaborative elaboration and discussion of a pedagogical approach mediated by the use of videos to the education of the citizen as a consumer human being? (Carvalho, 1999).

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STUDENTS' INTERPRETATIONS OF GRAPHICAL REPRESENTATIONS INVOLVING CHANGING SPEED

Roberta Y. Schorr and Gerald A. Goldin

Rutgers University, New Jersey, USA

At Rutgers University, we are engaged in a long-term research and implementation study with SimCalc (Kaput and Roschelle, 1997), where disadvantaged, inner-city high school students are provided with technological tools intended to develop their understandings of the conceptual building blocks of calculus. The overall goal of the study is to understand how best to help such students learn essential mathematical ideas, making use of a technology-based environment in which phenomena can both be created for students to experience, and represented for students to manipulate and discuss. The particular analysis presented here addresses the question of cognitive obstacles associated with graphical representation of changing speed vs. time. We have chosen this because graphical representation of position vs. time, and velocity vs. time, is a fundamental tool in calculus, where the notion of velocity comes to embody the idea of an instantaneous rate of change of position or displacement with time, and research has shown that students frequently experience great difficulty with the interpretation of graphical representations and problems involving the concept of speed (Monk and Nemirovsky, 1994; Thompson, 1994).

The students with whom we are working show some evidence of cognitive obstacles similar to those previously inferred in children, college students, and adults. However, the rich structure of linked external representations provides a valuable experimental context for exploring and overcoming these obstacles. We will discuss our observations in the context of expectations based on theories of internal and external representations and their development.

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SINGLE-DIGIT MULTIPLICATION PERFORMANCE IN JAPANESE ADULTS: ASSESSING THE PROBLEM SIZE EFFECT

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1. Introduction

Cognitive approach to simple computation have developed over about 30 years. One of the most interesting phenomenon is "problem size effect", which generally means that small operands problems is solved faster than large, for example, " 3 × 4 " is solved faster than " 6 × 7 ".

Many researches and experiments have been made, but almost all experiments made in Europe, Canada and America. I think that Japanese show different pattern of problem size effect because of differences of educational programs. The purpose of this research is to examine single-digit multiplication performance, and to show pattern of problem size effect in Japanese adult.

2. Experiment

Procedure: 12 University students participated in this study. They solved the 100 multiplication and addition problems(addition was excluded from this analysis) which consisted of the combinations of all single-digit(0-9). Problems were presented on a video monitor. When problems were presented, participants were required to make oral answers, and to press the key on starting to make answers. These answers were recorded by experimenter in order to judge errors, and RT(Reaction time) which is the time participants took from presenting problem to pressing the key was counted and recorded by computer.

Result: Figure shows mean RT for each operand family. The effects of problem size, 5-operand and tie occured. These effects matched many researches.

3. Conclusion and Discussion

Canadian and Japanese shows different patterns for large problems in Figure(data of Canadian is by LeFerve et al., 1997, and note that they used voice key). The larger size of problems was, the larger differences were. Chinese was similer to Japanese (LeFevre et al., 1997). I expected that this phenomenon ocuured because of differences of educational programs. In Japan, children learn multiplication at 2-grade, and all of them are requred to memorize multiplication table by recitation. For example, children memorize "7 × 9 = 63" by saying "Shichiku rokujuu-san". So almost all adults from Japan probably solve large size problems by recitation procedure.

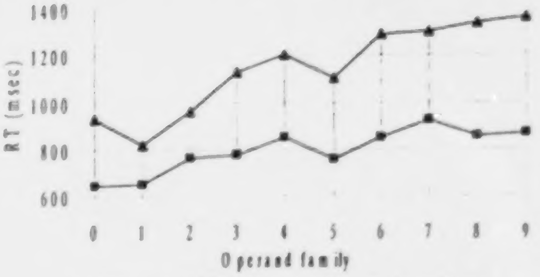

Figure: Mean RT for operand family (note that the data of Canadian is by LeFevre et al.,1997)

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Proposal to Junior High School Mathematics Education Based on "The Third International Mathematics and Science Study"

Siro SUWAKI
Okayama University of Science

We have long appreciated the seriousness of the problem that more and more pupils dislike maths and science. By using the data of "The Third International Mathematics and Science Study", I analyzed the factors to cause them to like maths both from the pupil viewpoint and from the teacher viewpoint. Based on the analysis, I would like to propose the following as a solution to the problem.

1. Generally, when teachers force pupils to reach high levels of accomplishment in maths, their scholastic abilities are surely improved. On the other hand, they tend to dislike it. As in Singapore, however, with an earnest effort of the government and teachers, it is possible to maintain a high level of scholastic ability in maths and hold pupils' interest in it at the same time.

2. The more homework and guizzes, the more pupiles tend to come to like maths , though they should not be excessive. Therefore, teachers must not hesitate to give their pupils homework.

3. Pupils who are interested in "proportion" in maths tend to like maths. Teachers should make use of it in order to make their pupils interest in maths. It is advisable to cite many examples of "proportion" in actual life.

4. Pupils who find maths easy and interesting, consider it important in life, and seek work using it tend to like maths. Teachers should keep it in mind in class.

5. The clearer the pupils' motivation to have good marks, the more they like maths. Of all the pupils from many countries, Japanese ones have the least motivation that they want to make their parents happy. Perhaps family affection is necessary to enhance pupils' motivation to do well in the test.

6. Pupils who can explain the reason for their own answers tend to like maths. As often as opportunity allows, teachers should encourage their pupils to express themselves without being afraid of making mistakes.

7. When teachers decide how to teach under the guidance of the Ministery of Education, their pupils tend to like maths. Therefore, besides textbooks, teachers should make the best use of the manual by the Ministery of Education.

8. Pupils give strong support to evaluations of their homework. Homework is to establish a bond between teachers and pupils, so it is necessary to give and grade homework.

FUNCTIONAL RELATIONS AMONG INTERNAL REPRESENTATIONS OF MATHEMATICAL WORD PROBLEMS

Atsumi Ueda
Hiroshima University

For the last few decades, mathematics educators have been concerned with problem solving. Many theories related to problem solving have regarded it as a process of refining internal representation of problems to be solved. The concept of internal representation plays an important role in explaining the essential parts of problem solving.

Generally, if we attempt to realize a phenomenon as a refining process of some related aspects, we have to make clear the essential characteristics of its main aspect and describe its refining process by some way. We need to explain what is the internal representation of a problem and contrive some ways to grasp its refining process operationally. The latter is the focus of this proposal.

Silver (1979) investigated students' perception of mathematical problem similarity by using the Card-Sorting Task (CST). According to the principle of "second-order isomorphism" (Shepard & Chipman, 1970), CST is certainly a kind of convenient way to seek the structure within the relations among internal representations of mathematical problems. But it may not be able to get the information about internal representation from CST, because the problems that are classified into the same group have to be regarded as having made the identical representation.

In the present study, multidimensional scaling was used as a class of effective statistical procedure for the purpose of seeking the structure of the relations among internal representations of mathematical word problems. Possible effectiveness of this kind of procedure will be presented by using the pictorial configuration that indicates the structural relations among internal representations.

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ENRICHMENT FOR MATHEMATICALLY PROMISING STUDENTS IN THE UK

Jenni Way

University of Cambridge and Royal Institution of Great Britain

Provision for talented young mathematicians in the United Kingdom takes a variety of forms. Two complimentary national programmes have arisen out of the generous sponsorship of several organisations that recognise the need to nurture the abilities of school students who exhibit enthusiasm and/or high ability in mathematics.

NRICH Online Maths Club http://nrich.maths.org.uk

One of these programmes, based at the University of Cambridge, is a website that provides mathematical puzzles, problems, games, articles and news for students aged 5 to 16 years and their teachers, free of charge. New material is made available on the first day of each month and students are encouraged to send in solutions and other items for publication. There are also well utilised communication services, such as the Ask-a-Mathematician service and a discussion web-board. The website has in excess of 40 000 regular users, has 4 334 registered members from 73 countries and receives about 10 000 hits per day. About 25% of children are accessing the site from home.

The poster will show a screen shot of the website and a graphical display of some site statistics. Brochures outlined the website's content will be made available.

Royal Institution Masterclasses

The other programme consists of a national network of mathematics 'masterclasses', supported by the Royal Institution (Registered Charity 227938), but run by local volunteers. The secondary-level network, catering for students of about 15 years of age, began 20 years ago and currently consists of 40 groups, running 60 series. Schools are invited to nominate students to attend a series of Saturday morning classes, run by volunteer teachers and guest lecturers. Participants are challenged by problems and tasks on a particular topic, and encouraged to interact with other students. Some groups run special summer camps or programmes.

In the latter half of 1998, a similar network was by initiated the Royal Institution for Primary school age children. A wide range of organisational models has developed in response to local needs and resources. Some classes are held on weekends, some during school time. Some groups work in partnership with secondary schools, some utilise the enthusiasm of pre-service teachers, others rely on various volunteers.

The poster will include some captioned photographs that illustrate some the attitudes and achievements of the 'masterclass' programmes.

CASE STUDY OF CONCEPTION ON LIMIT CONCERNING CHILDREN

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10 age children in japanese experimental elementary school attached to Univ. could find density of rational number. They found that there are many fractions for instance 5/12, 7/18, 8/18, 9/24, 10/24, 11/24, next 4 fractions, after next will emerge 5 fractions soon after immeasurable, between two different fractions 1/2 1/3.

It was the learning that teacher assigns subject matter, then pupils advance it. This result of learning is in agreement with report of the Cambridge Conference on School Mathematics thet pupils in grade 3-6 begin to consider infinite sequences of real number. Piagte and Inhelder examined that the child at age of 8 reaches four hundred points at age of 10 suggeste "nine thousand points, may be even more" for a 1cm. square.

Note on Mathematics in Primary School(Cambrige1969), gave a description that the mathematical experiences of a child before the age of eleven and responses determine his potential mathematical development.

In this essentials I choose divergent problem. Stanley Tabach gave me much of information. The result is shown for 8, 10 age level in Table 1.

Age	task Divergent square	degree of understanding concrete level(%)			degree of understanding abstract level (%)		
		*	**	***	*	**	***
8		70	20		20	36	36
		(72)			(28)	(20)	(24)
10		80	16		48	36	12
		(80)			(64)	(12)	(12)

Table 1 *clear **uncertain ***none ( ) result of S.T.

The subject of this study are pupils of elementary school in average area.

10year-old subject (80) were understand on the concrete level,and 48% on the abstract level. This is remarkable result, I think.

In japan Ministry of Education distributed new course of study in 1999. It emphasized the importance of doing research on mathematical activity.

Teater gives subject, then pupils can advance. Up to now that would not have been allow to advance beyond grades except experimental school. This report is worth understanding stage of pupils for teaching.

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VISUALIZATION OF THE PRODUCT OF COMPLEX NUMBERS IN THE TEACHING

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The problem on teaching of complex number in high school mathematics is old and new. The reasons of difficulty to teach a complex number would be caused by its abstract character, that is, the ambiguous expression of complex number and also this number is the one of 2-dimensional. To visualize the complex number and its compositions, the complex number plane is introduced. Then, a complex number is a vector or point in this plane. The sum is the fourth point of parallelogram formed from two number vectors.

The purpose of this paper is to visualize elementary the product in this plane from the viewpoint of $|zw| = |z|\ |w|$ ...(1) instead of the polar form of complex number, where $|\ |$ denotes the absolute value of complex number. It is well known that this relation characterizes the complex number in the orthogonal coordinate plane with product satisfying some conditions.

At first, call a+bi, $i^2 = -1$, a complex number. Define the equivalence, sum, and product as usual. Next, introduce the 2-dimensional real vector space with inner product or the orthogonal coordinate plane. For complex numbers $z=a+bi$, $w=c+di$, the product zw is constructed as a sum of vectors $a(c,d)+b(-d,c)$. From the known identity $(ac-bd)^2+(ad+bc)^2=(a^2+b^2)(c^2+d^2)$, which is equivalent to the relation (1), we have a usual construction of the product as a rotation about the origin and an extension of vector. The relation (1) and the Pythagorean theorem imply the addition theorem of trigonometric function. The roots of a quadratic equation are taught as the intersection of the parabola and the x-axis if the discriminant is non-negative and the one of the axis of parabola and a certain circle with the origin as center if the discriminant is negative. We believe that such visual explanation of the product would assists student for his understanding of complex number and its properties.

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SCHOOL VISIT

Brief Guidance to Japanese Mathematics Education

Tadao Nakahara Hiroshima University
Toshiakira Fujii Tokyo Gakugei University
Masataka Koyama Hiroshima University

1. Educational Framework

After the end of World War II, the educational system in Japan has was reorganized, both in structure and curriculum. The so-called 6-3-3-4 system of schooling has been implemented and fundamentally continued up to now. It is essentially composed of six-year Elementary School, three-year Junior High School, three-year Senior High School and four-year of university. These schools and universities can be classified as national, local public or private.

The first two levels - Elementary and Junior High- are compulsory education. Children usually start their formal education at 6 years old and change from Elementary to Junior High school at age 12 years. Unlike in many other countries, the school year in Japan begins on 1st April and ends on 31st March of the following year.

The basic school system is summarized in Figure 1.

Age	Grade	Type of School
18-22		University
17-18	12	Senior High School
16-17	11	
15-16	10	
14-15	9	Junior High School
13-14	8	
12-13	7	
11-12	6	Elementary School
10-11	5	
9-10	4	
8-9	3	
7-8	2	
6-7	1	
3-6	K	Kindergarten

←Entrance Examination (between Senior High School and University)

←Entrance Examination (between Junior High School and Senior High School)

Compulsory Education (Grades 1-9)

Figure 1. Educational System in Japan

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2. Changes of Mathematics Curriculum

The basic framework for school curricula is outlined in the Course of Study issued by the Ministry of Education. It is the aim of the Course of Study to help ensure that an optimum level of teaching and learning be secured in all Elementary, Junior High and Senior High Schools, based on the principle of equal educational opportunity for all. A Course of Study has been revised approximately every ten years. We could point out the following as main features of mathematics education based on each course of study for mathematics.

(1) Life-unit Learning (from 1947 to 1958)

The Course of Study in this time was made under the strong guidance of the American educational mission. The so-called "Life-Unit-Leaning" began to be implemented. The goal of life-unit-learning was to learn how to use mathematics in every day life. This curriculum was severely criticized, because the level of children's performance in mathematics had dropped.

(2) Systematic Learning (from 1958 to 1968)

Japan changed the Course of Study from life-unit-learning to studying mathematics systematically in 1958. The level of content to be learned became higher.

(3) Modernization (from 1968 to 1977)

The Course of Study in this time was made according to the direction of the international movement of Modernization of Mathematics Education. For example, the concept and symbols of "set" were introduced and pure mathematics was emphasized. However, many children could not understand New Math, so the mass communication, parents and some mathematicians criticized this curriculum.

(4) Back to Basics (from 1977 to 1989)

This Course of Study was influenced by many criticisms to the results of modernization and was characterized as Modification of Modernization. The basic contents of mathematics were emphasized, so the level of mathematical content was pulled down.

(5) Integration of Cognitive and Affective Aspects (from 1989 to 1999)

The Course of Study for mathematics education was revised toward integration cognitive and affective aspects in 1989. For example, the following objective was set up in elementary level. "To help children develop their abilities to consider daily-life problems insightfully and logically, and thereby foster their attitudes to appreciate the mathematical coping with and to willingly make use of it in their lives.

(6) Latitude through Intensive Selection of Educational Content (from 1999)

During these ten years, such problems as unschooling and "classroom in crisis" have become quite notable and they are attributed to excessively stressed life of children. Therefore the Course of Study was revised and its educational contents were slimmed down intensively. The 30% of them was eliminated from the Elementary and Junior High School level.

The revised Course of Study put into effect in April, 2002 for all grades in Elementary School, 2003 for all grades in Junior High School, 2004 for the first grade in Senior High School, and so on.

3. Teaching and Learning in Mathematics Education

Throughout the Elementary phase of education, mathematics teaching is inclined towards a child-centered and problem-solving approach. For example, a typical 45 minute lesson will follow the pattern shown below.

Minutes	
0	⟨Children's standing up and bowing⟩
	⟨Reviewing the last lesson or presenting some familiar topic⟩
5	⟨Understanding a problem⟩ The teacher presents a problem which contains mathematical concepts, facts and skills. Children try to understand it.
10	⟨Solving the problem for themselves⟩ Each children solves the problem individually. The teacher encourages their children to solve the problem for themselves.
25	⟨Reporting their solutions⟩ The teacher asks several children to write their solution on the blackboard and explain their way of thinking in solving.
30	⟨Discussion of the solutions⟩ Children discuss and compare their solutions with help of the teacher, and find common ideas or a refined solution.
40	⟨Summing up by teacher⟩ The teacher summarizes the day's mathematical ideas, facts and skills.
45	⟨Children's standing up and bowing⟩

Teachers usually use a blackboard to record a lesson. Two parts are stressed: one is the thinking processes and the other is the important mathematical thinking. Investigations and practical work have been encouraged, particularly for children in Elementary School.

In Junior High School, the teaching style may be the one that the teacher explains the mathematical concepts and skills by using examples, manipulative aids and published resources, and gives enough exercises for pupils to become skillful in computation. In mathematics teaching for Senior High School, a more traditional style of teaching is normally adopted.

4. Assessment in Mathematics Education

In Japan, there exist the national curricula as we mention above, but the national tests for assessment do not exist. Many teachers appreciate, in general, the importance of formative evaluation, but they also tend to depend on summative evaluation using paper and pencil tests. Teachers frequently give brief tests to ascertain and control achievement levels of learners. They will try their best to help pupils acquire knowledge and use skills.

Formally, there are two aspects of evaluation: an academic evaluation and the four additional points of evaluation. The former designates 5 (the most upper level) to 1 (the lowest level) scales to denote the performance of each pupil. The four different viewpoints of the latter aspects are:

1. Interest / Willingness / Attitude,
2. Mathematical thinking,
3. Representation / Processing,
4. Knowledge / Understanding.

In Japan, the entrance examination of the Senior High School is the first external examination of its kind given to pupils. Its public examinations are set by each Prefecture. However, each private Elementary and High School sets also its own entrance examination. The examination for entrance to a national university is usually taken at the age of 18 and above. This takes two distinct forms. The so-called center examination is the common test given for all candidates. The other is administered by each national university. Each private university usually sets its own entrance examination.

5. Teacher Training Routes

5.1. High School Mathematics Teachers

There are two main routes:

(1) 4-year undergraduate course in Mathematics Education, with substantial periods of teaching practice, at Faculty of Education in a university. Students in this course are

awarded a B.Ed. degree leading to a teaching certificate (First Class Certificate) when they have completed their course by obtaining the required number of credits including mathematics, mathematics education, didactics, psychology, etc.

(2) 4-year undergraduate course in Mathematics at a Faculty of Science or the like. Most Students on this course are awarded a B.Sc. degree and get a teaching certificate (First Class Certificate) when they have obtained the required number of credits including mathematics, mathematics education, didactics, psychology, teaching practice, etc.

Moreover, there is a 2-year postgraduate course for those students with such suitable qualifications as a B.Ed. or B.Sc. degree and a First Class teaching certificate. Students on this course are awarded a M.Ed. or M.Sc. degree leading to a teaching certificate (Advanced Certificate) when they have obtained the required number of credits and passed the final examination based on a submitted Master's thesis.

5.2. Elementary Teachers

There are two main routes:

(1) 4-year undergraduate course at Faculty of Education or the like, leading to a B.Ed. degree and a teaching certificate (First Class Certificate). There will be a major mathematics component for all students in this course, with extra courses for some students intending to specialize in mathematics.

(2) 2-year course provided at a junior college. Students on this course are awarded a teaching certificate (Second Class Certificate) when they have completed their course by obtaining the required number of credits.

Moreover, there is a 2-year postgraduate course for those students with such suitable qualifications as a B.Ed. degree and a First Class teaching certificate. Students in this course are awarded a M.Ed. degree leading to a teaching certificate (Advanced Certificate) when they have obtained the required number of credits and passed the final examination based on a submitted Master's thesis.

There is also a part-time route and correspondence courses in education for those students who have graduated a junior college or a university without a teaching certificate.

6. Issues of Concern

It is sure that the average score of Japanese students' achievement in mathematical knowledge and skill is relatively high. However, as we mentioned above, the educational contents are slimmed down in the new Course of Study. Therefore, we are worrying that the level of children's mathematical abilities will drop. And many students at High School level do have a negative attitude toward mathematics. So, we must endeavor to foster children's positive attitude toward mathematics. On the other hand, a new direction in the aim of mathematics education are proposed as "development of foundation for creativity". This is very important aim. So we are expected to realize this aim in mathematics education.

These situation will force mathematics teachers to reflect seriously on their educational philosophy and teaching methods. The teacher training courses at universities will be also change to meet those needs.

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Lesson Plan (1st Grade)

Teacher: Yasushi Miyamoto

0. Date, Place, Class

Date: 25th July, 2000,Tuesday (13:40 - 14:30)
Place: Elementary School attached to Hiroshima University, Music Room
Class: 1st Grade(Elementary School), 40 pupils (20 boys and 20 girls)

1. Teaching Unit: Subtraction

2. Teaching Objectives and Teaching Schedule

(1) Teaching Objective

Pupils should be able to understand the meaning of subtraction through various situations to be handled with subtractions.

(2) Teaching Schedule (8 hours as a whole)

Step 1: To think about a method of subtraction that is to find a complement of a number
5 hours

Step 2: To think about a type of subtraction that is to find the difference of two numbers
3 hours (Today's lesson is the 1st of 3.)

3. About Teaching Material

The pupils in the class have been learning the composition of numbers up to ten by using some objects, models and number cards etc. Besides they have been learning relations between two subsets and a universal set and some conditions of addition in the way of combining two amounts into one with numbers up to ten.

I want to set activities to think about the conditions for using subtraction and the meaning of subtraction in a concrete situation since the pupils are getting ready to consider compositions and decompositions of numbers up to ten. For that purpose I encourage the pupils to examine meanings of two different types of subtraction i.e. to find a complement and to find a difference as I put an emphasis on their operations to compose and decompose numbers by using concrete objects.

In this lesson I introduce the topic with a different number of examples from the ones they used in the previous lessons, and then I encourage them to think about the target problem of this lesson. After that I direct them to pay attention to the difference from the problems / operations of subtraction given in the previous lessons so that they discuss their aim of the lesson and think about correspondence between the expressions and the operations.

At the stage of practical operations, I direct their attention to the difference of the operations, such as operation for moving which they have been using by now, and operation for correspondence or operation for coupling which are expected to be used in this lesson, and encourage them to think about expressions to get the correct answers.

4. Problems to be Offered

The pupils should pay attention to the differences of the situations and of the operations of *Ohajiki*; (small disc), and work out the expressions to find a difference, and think that, as an expression, the two operations, i.e. to find a complement and to find a difference, come to the same subtraction, the same expression and the same answer. That leads them to the idea that the two types of subtraction can be integrated into one expression.

The comparison of the two subtractions that have different operations should be emphasized so that the integration of the meanings of subtraction would be discovered by the pupils.

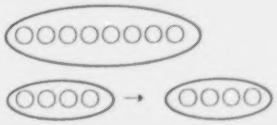
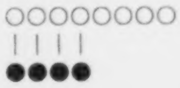
5.Today's Lesson

(1) Title of the Lesson: Explanation of subtraction by using operations.

(2) Objective of the Lesson

Pupils should be able to consider the difference and the common characteristics of the two types of subtraction; to find a difference and to find a complement.

(3) Teaching and Learning Process of the Lesson

Learning Activities	Intention and Process of Teaching	Points of Evaluation
1. Setting up a task (1) Discuss the problem of the lesson.	There are 8 white flowers and 4 red ones. ● Discuss the problem the pupils can pursue from the given situation. ● Pay attention to the question, "Which is and how many more?"	● Can the pupils pose the problem by themselves?
(2) Make sure the aim of the lesson is understood by the whole class.	Try to explain how to calculate. ● Think the situation by using *Ohajiki* or blocks. ● Pay attention to the difference between how to find a complement and to find a difference.	● Do the pupils understand the aim? ● Can the pupils explain the operation with the objects?
2. Investigating the task (1) Examine expressions and answers by using the objects.	Explanation with an operation of moving. 	● Can the pupils get the expression and the answer?
(2) Integrate the ideas by mutual consent.	Explanation with an operation of correspondence. 	● Do the pupils pay attention to the difference from the topics in the previous lessons?
3. Developing the task (1) Discuss the problems for the next lesson.	● Summarize what the pupils noticed from the difference of the operations of the objects and from the fact that both operations come to the same expression and answer. ● Discuss what comes in the next lesson.	● Do the pupils understand the problems in the next lesson?

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Lesson Plan (4th Grade)

Teacher: Takeshi Nakamura

0. Date, Place, Class

Date: 25th July, 2000,Tuesday (13:40 – 14:30)
Place: Elementary School attached to Hiroshima University, Special Room(2)
Class: 4th Grade(Elementary School), 40 pupils (20 boys and 20 girls)

1. Teaching Unit: Division of Whole Numbers

2. Teaching Objectives and Teaching Schedule

(1)Teaching Objective

Pupils should be able to understand how to operate divisions by 2-digit divisors and calculate them properly.

(2)Teaching Schedule (12 hours as a whole)

Step 1: Division by a 2-digit number (Part 1)	6 hours (Today's lesson is the 2nd of 6.)
Step 2: Division by a 2-digit number (Part 2)	3 hours
Step 3: Rules of division	3 hours

3. About Teaching Material

The pupils in the class have been learning the meanings and operations of division of a 2-digit number by a single digit number. Therefore this topic aims to expand the range of numbers for the calculation and seek the proper operation for division of a 2 or 3-digit number by a 2-digit number. Besides it aims to cultivate ideas of units and those of correspondence, and also to deepen the students' ideas and senses about numbers.

At first, I assign the pupils a calculation such as 80÷20 and direct the pupils' attention to a unit of ten which enables them to apply *Ku-Ku*; a multiplication table of single digit numbers, to the calculation with some operations of objects like some blocks. Secondly, the pupils are encouraged to verbalize their aims by highlighting the difference from the divisions they used in the previous lessons. The pupils are to notice that *Ku-Ku* is applicable to the calculation when we consider place values and utilize approximation.

Furthermore, the pupils should characterize each way of calculations while they discuss and examine each process of the calculations so that they can find out the ways that they can be applied to various cases of division. Hopefully the pupils can understand the advantages of mathematical thinking and operations in their daily life through this kind of work, and they can improve their ideas and attitudes to look for new problems and develop them.

4. Problems to be Offered

I want to derive various types of calculation from the pupils' activities by asking for the ways of division of 84÷21. And then I encourage the pupils to discuss new problems to operate divisions while calculating some divisions of a 2-digit number by a 2-digit number in as many ways as they can find.

5.Today's Lesson

(1) Title of the Lesson: Division of 2-digit numbers.

(2) Objectives of the Lesson

Pupils should be able to solve problems by calculating various kinds of divisions by exploring the way of divisions of 2-digit numbers

(3) Teaching and Learning Process of the Lesson:

Learning Activities	Intention and Process of Teaching	Points of Evaluation
1. Setting up a task		
(1) Discuss the problem of the lesson.	We have 84 pencils. How many people get pencils when we distribute them 21 pencils each?	● Do the pupils try to make their problem for themselves?
(2) Identify the aim of the lesson.	● Focus the problem in comparison with the question, "80÷20," which is given in the previous lesson.	● Do the pupils catch the aim?
2. Investigating the task		
(1) Think about how to calculate the problem.	How can we work out a division of 2-digit numbers?	
	● Seek different ways for calculating the problem so that each pupil has his/her own idea.	● Do the pupils express their own ideas clearly?
	● Examine their ideas with practical activities or multiplications.	● Do the pupils verify their ideas?
(2) Present individual ideas to the class.	Calculate every place separately. $80 \div 20 = 4$ $4 \div 1 = 4$ Use approximate numbers. $80 \div 20 = \boxed{4}$ $21 \times 4 = 84$	
(3) Observe ideas from the others and express own ideas.	● Think about advantages of the ideas given by the classmates by comparing them with their own ideas. ● Think about the exception which we can not apply in the usual way.	● Do the pupils construct and express their opinions related to the other opinions?
3. Developing the task		
(1) Discuss the problems for the next lesson.	● Think about the problems for the next lesson through various divisions to know the problem considered to find an interim quotient,.	● Do pupils catch the problems in the next lesson.
(2) Conclude the lesson.	● Review the whole work of the lesson.	

Lesson Plan (5th Grade)

Teacher: Toshiyuki Akai

0.Date,Place,Class

Date: 25th July, 2000,Tuesday (13:40 - 14:30)
Place: Elementary School attached to Hiroshima University, Special Room(1)
Class: 5th Grade(Elementary School), 39 pupils (19 boys and 20 girls)

1.Teaching Unit: Areas of Figures

2.Teaching Objectives and Teaching Schedule

(1) Teaching Objective

Children can find areas of triangles and quadrilaterals by making use of the area of rectangle they learned previously.

(2) Teaching Schedule (14 hours as a whole)

Step 1: Ways of finding areas	1 hour	(Today's lesson)
Step 2: Areas of quadrilaterals	5 hours	
Step 3: Areas of triangles	4 hours	
Step 4: Elaboration of ways of finding areas	4 hours	

3.About Teaching Material

The objective of this teaching unit is to let children find areas of triangles and quadrilaterals by reducing the areas to those of rectangles and squares which they have learned previously.

Children learned about areas of squares and rectangles when they were in grade 4. Children also learned, through activities such as tessellations, that a figure can be seen as different figures by moving parts of it and looking at it from various angles. In grade 5, it is desired that children make use of these previous learning experiences and come up with ways of changing novel figures into rectangles, such as moving parts of parallelograms and rhombuses without changing their areas, or putting together two congruent triangles.

In the teaching, to encourage children's spontaneous development of different ways, I plan an activity of making quadrilaterals by 4 congruent right triangles. In the activity, I want children to recognize that they can change the quadrilaterals into rectangles and make use of previous learning. In doing so, I want children to focus on the question "which lengths do we need to know in order to calculate the area by making use of the area of rectangles," which will lead them to formulas for areas of parallelograms and so on. I also want to utilize this idea to find areas of triangles.

4.Problems to be Offered

Make various quadrilaterals by using 4 congruent right triangles. Then think about ways of finding the areas of quadrilaterals made based on the right triangles.

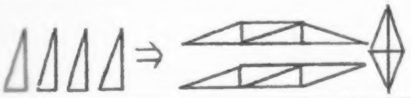
5.Today's Lesson

(1) Title of the Lesson: Making quadrilaterals from right triangles and find the areas

(2) Objective of the Lesson:

Children can think about ways of finding areas of quadrilaterals based on the movement of right triangles.

(3) Teaching and Learning Process of the Lesson:

Learning Activity	Intention and Process of Teaching	Points of Evaluation
1. Setting a Task (1) To make figures by 4 congruent right triangles.	● Let children make figures by 4 congruent right triangles. Make figures by using 4 congruent right triangles, and think about the ways of finding areas of these figures. 	● Can they make figures learned previously? ● Can they grasp the task?
(2) To find areas.	● Let children think about areas of the figures they made.	● Can they recall the way of finding areas of rectangles?
2. Investigating the Task (1) To think about the ways of finding areas.	● Let children recall rectangles, of which they can find areas. In what way can we find the areas?	
(2)To change the figures into rectangles.	● Let children change the figures into rectangles. ● Let children think about the ways of finding areas by calculation.	● Can they change the figures into rectangles? ● Can they find the lengths, which are needed to find the areas?
3. Developing the Task (1) To think about the ways of finding areas of various figures.	● Let children imagine various ways of finding areas of parallelograms and triangles.	● Do they know the task for the next lesson?

Lesson Plan (7th Grade)

Teacher: Atsushi Nagao

0.Date, Place, Class

Date: 25th July ,2000, Tuesday (13:40-14:30)

Place: Junior and Senior High School attached to Hiroshima University,
Kenshu-kan #3 Room

Class: 7th Grade (Junior High School, 1st grade), 40 students (20 boys and 20 girls)

1.Teaching Unit: Positive Numbers and Negative Numbers

2.Teaching Objectives and Teaching Schedule

(1) Teaching Objectives

1. Using positive and negative numbers in order to show their complementary relationship
2. Able to do addition and subtraction of positive and negative numbers
3. Able to do multiplication and division of positive and negative numbers
4. Able to use four operations with positive and negative numbers
5. Convert fractions to decimal numbers in order to deepen students' interest toward mathematics

(2) Teaching Schedule (10 hours as a whole)

Step 1: Positive numbers and negative numbers	2 hours	
Step 2: Addition and subtraction	3 hours	
Step 3: Multiplication and division	4 hours	
Step 4: Advanced work (fractions and decimals)	1 hour	(Today's lesson)

3.About Teaching Material

For the Special Activity in PME24, I have planned an extra 1 hour class period for some advanced work on Positive and Negative Numbers. The content is based on my own personal experience. When I was in elementary school, I learned how to convert fractions to decimal numbers: by dividing the numerator by the denominator and found that the answer could be either a finite answer or a recurring decimal. However, it was only when I become a teacher 20 years ago, that I thought about the reason for this. Few students do pay attention to the facts, such as "the answer should be finite only when the denominator is a multiple of 2 or 5 (which are the measures of 10)", "an answer becomes recurring when the remainder is smaller than divisor, and finite", and so on. Through this lesson, I intend to get students to reconsider the numbers which they have learnt, as well as their ways of studying mathematics.

Although few students in this class hate mathematics, not many of them tend to pursue mathematical ideas enthusiastically and aggressively. I hope this lesson will help students change their attitudes toward mathematics in this way.

4.Today's Lesson

(1) Title of the Lesson: Fraction and Decimal Numbers

(2) Objective of the Lesson: Get student to think their answers and reason "which numbers would be finite or recurring when they convert fractions to decimal numbers."

(3) Teaching and Learning Process of the Lesson

Teaching Content	Teaching Process	Notes
1. Introduction Presenting the Problem	Have the students choose any irreducible fraction, and get them to convert this fraction to a decimal number.	Find out whether the students know how to convert to decimal numbers.
	Ask some students to write their numbers on the blackboard.	
	Identify the kinds of decimal numbers obtained when converting from fractions.	If necessary, introduce the terms "finite" and "recurring".
2. Development Posing Task 1	What kinds of fractions would be finite? Why?	
Solving Task 1	Students discuss in small groups, and present their ideas.	The groups will be 4 to 6 people, with neighbors.
	Discuss the presented ideas. Questions, any other ideas, and comments.	Take plenty of time.
	Summarize the discussion, and confirm the correctness of their reasons for obtaining decimals.	Ask repeatedly for confirmation of results.
Posing Task 2	If the answer is not finite, why does it become recurring?	
Solving Task 2	Give students opportunity to make comments.	Make optimal use of concrete examples.
	Discuss the comments, and deepen their ideas.	
	Make sure reasons are given and understood for the presence of recurring decimals.	
3. Summary Summary of the lesson and students' impressions.	Summarize the day's topic, and ask some students to make any comments on the lesson.	

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Lesson Plan (9th Grade)

Teacher: Yasuhiro Inosako

0. Date, Place, Class

Date: 25th July, 2000, Tuesday (13:30 – 14:30)

Place: Junior and Senior High School attached to Hiroshima University,
Information Processing Room, Information Processing Center

Class: 9th Grade(Junior High School, 3rd grade), 40students (20 boys, 20 girls)

1. Teaching Unit: Properties of Circles

2. Teaching Objectives and Teaching Schedule

(1) Teaching Objectives

1. To help students: investigate the properties of chords and tangents of a circle by paying attention to the symmetry of circles, understand the properties of inscribed circle of a triangle or of a quadrilateral, and deepen the understanding of the relation between the positions of two circles.
2. To help students understand the relation between the angle at the circumference and the one at the center of a circle and expand their ability to prove the properties of circles by using this relation.
3. To help students understand the properties of a cyclic quadrilateral and the condition that four vertices of a quadrilateral are on a circumference with the relation between the angle at a circumference and the one at the center.
4. To help students understand and grasp synthetically the relation of the angle between a tangent to a circle and a chord through the point of contact and the angle subtended by the chord at a circumference with the theorem which has already been learned.

(2) Teaching Schedule (11 hours as a whole)

Step 1: Properties of circles	4 hours	
Step 2: Angle at the circumference	4 hours	(Today's lesson is the 1st of 4.)
Step 3: Circles and Quadrilaterals	3 lessons	

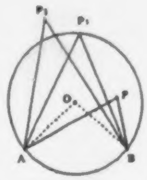
3. About Teaching Material

Students have already learned geometry from the construction of basic figures and solid geometry in the 1st year of Junior High School (7th grade), and the conditions for congruent triangles or similar triangles and the properties of parallelograms in the 2nd year (8th grade). In the 3rd year in the Junior High School (9th grade), the aim is to expand the students' ability to measure figures by making comprehensive use of the properties and theorems concerning cyclic quadrilaterals and Pythagoras' theorem.

In order to help students understand the properties of figures clearly, the students should first surmise discover and make sure of the properties of a particular figure, and then build up theorems with understanding of the necessity and the validity of definitions. Such a teaching method has not been easy so far, but the introduction of simulation software has made it possible and easier. This lesson is intended to get students to surmise the relation between the angle at the circumference and the one at the center of a circle, and to examine the cases where the vertex is inside the circle or outside of it, and to understand all these cases synthetically.

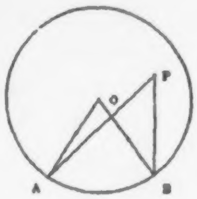
4. Today's Lesson

(1) Title of the Lesson: Surmise the relation between the angles at the circumference and at the center of a circle



(2) Objective of the Lesson: There is a circle, center O, and its sector OAB on a plane. To help students observe the change in size of $\angle APB$, take any point P on plane. They should then examine the relationship between the size of $\angle APB$, that of the angle at the circumference and the position of the point P, and attempt to understand all these cases synthetically.

(3) Teaching and Learning Process of the Lesson

Teaching Content	Teaching Process	Notes
1. Introduction Presentation of problem	· The problem of this lesson There is a circle, center O, and its sector OAB shown as the right figure. Move the point P on the plane, examine the size of $\angle APB$. What can be found about it?	The software indicates the size of $\angle APB$ with fixed points, A and B, and point P moving freely. 
2. Development To examine the size of angle to solve the problem.	1. To observe the size of angle APB and to surmise the possible property by changing the position of point P with the following procedure. To examine the case where the size of different angles ABP are equal. To examine the case that the point P is in the circle or out of circle. To move the points, A and B, if necessary.	Take arc AB as a minor arc
Confirmation of the properties Consideration for summary	2. To present the discovered properties. 1)When the point P is at the circumference, $\angle APB$=const.(α) 2) α is a half of the angle subtended by the minor arc AB at the center. 3)When the point P is in the circle, $\angle APB > \alpha$ 4)When the point P is out of the circle, $\angle APB < \alpha$ 5)It is impossible to sum up shown as from 1) to 4) in case that the point P is in the opposite side to the point O with respect to the line AB.	2) should be found by students' spontaneous thinking. When the point P is in the opposite side to the point O, help students notice the following points. · Which angle should be examined, the obtuse one or the reflex one? · It is impossible to sum up the common properties etc.
3. Summary Confirmation and summary of this lesson	Summary of this lesson · To confirm that $\angle APB$ is called the angle subtended by the arc AB at the circumference based on the properties 1) and 2). · To confirm the discovered properties.	The assignment is whether the same properties can be found or not about major arc

Software: Cabri Geometry II for Windows (Texas Instruments)

Lesson Plan (10th Grade)

Teacher: Yoshihumi Inoue

0. Date, Place, Class

Date: 25th July, 2000, Tuesday (13:30 – 14:30)

Place: Junior and Senior High School attached to Hiroshima University,
Kenshu-kan #4 Room

Class: 10th Grade(Senior High School, 3rd grade), 40students (24 boys, 16 girls)

1. Teaching Unit: Quadratic Functions

2. Teaching Objectives and Teaching Schedule

(1) Teaching Objectives

1. To help students deepen their understanding of change and correspondence and grasp clearly the concept of function through problems.
2. To help students deepen their understanding of graph of general quadratic function, $y=ax^2+bx+c$.
3. To help students determine a quadratic function from conditions regarding graph.
4. To help students find out maximum and minimum values of quadratic function with help of graph.
5. To help students deepen their understanding of the relation between the graph of quadratic function and the solution of quadratic equation and inequality.
6. To help students solve various problems with help of quadratic function

(2) Teaching Schedule (26 hours as a whole)

Step1: Quadratic function and its graph	8 hours
Step2: Maximum and minimum of quadratic function	4 hours
Step3: Positional relation between graph of quadratic function and x axis	3 hours (Today's lesson is the 2nd of 3)
Step4: Quadratic function and quadratic equation	4 hours
Step5: Quadratic function and quadratic inequality	4 hours
Step6: Application of quadratic function	3 hours

3. About Teaching Material

Students have already learned about the function concept from the point of the relation between concrete variables in the domain, Quantitative Relations, in Junior High Schools. The content in the first year in the Senior High School is aimed at deepening students' understanding of functions and proceeding to the learning of quadratic function and its graph. One main objective here is to consider the maximum and minimum values of quadratic functions and their graphs and to solve problems using this knowledge.

Students can sometimes make a guess at an answer to geometrical problems by focusing on such aspects as symmetry, but often they are not sure about validity of their answers, when being asked a question "Is it really so?"

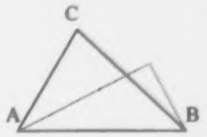
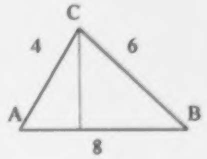
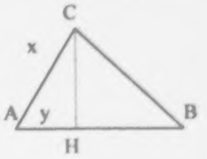
This lesson is aimed at getting students to intuitively guess an answer to geometrical problems and then to verify them. Students are expected to feel that mathematics can be a useful tool in justifying their guess.

4. Today's Lesson

(1) Title of the Lesson: Application of quadratic function

(2) Teaching Objective of the Lesson: To solve problems using quadratic function and appreciate the significance of a mathematical way of viewing and processing solutions.

(3) Teaching and Learning Process of this Lesson

Teaching Content	Teaching Process	Notes
1. Introduction Review	*Application of quadratic function · To confirm that various problems have been solved so far using quadratic functions and their graphs.	
2. Development Presentation of problem	*Presentation of problem The ends, A and B, of a 10 meters piece of rope are fixed at 8 meters apart. If any point C along the rope is taken and a triangle ABC is formed, find the position of C which results in the greatest possible area of triangle ABC.	
Guessing an answer and its verification	· To help students find at first the area of the triangle when AC=4 and BC=6. · To help students guess an answer to this problem and its reason.	· To help students have their own guess.
	*Maximum of the area · When both ∠A and ∠B are acute, a perpendicular line from vertex C is drawn onto the side AB and their intersection is named H. 1) To help students consider relation between x and y when AC=x and AH=y. 2) To express CH in terms of x. 3) To consider the case when CH becomes the biggest.	· To confirm that the area becomes the biggest when the length of CH becomes the biggest. 
3. Summary	· To consider the case when ∠A (or ∠B) is obtuse. · To conclude, as a result of consideration, that the area becomes the biggest when the figure is an isosceles triangle with AC=BC. *Summary of this lesson · To help students realize that the problem has been solved using quadratic function and to confirm practicability of function.	

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